STABILIZATION OF MULTIVARIABLE SYSTEMS WITH CONSTRAINED CONTROL STRUCTURE

A THESIS
SUBMITTED TO THE DEPARTMENT OF ELECTRICAL ENGINEERING
AND THE INSTITUTE OF GRADUATE STUDIES
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By
Kosur Alp Ünyelioglu
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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. M. E. Sezer(Principal Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. Özay Oral

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

STABILIZATION OF MULTIVARIABLE SYSTEMS WITH CONSTRAINED CONTROL STRUCTURE

Konur Alp Ünyelioğlu
M.S. in Electrical Engineering
Supervisor: Prof. Dr. M. E. Sezer
June 1988

The following problem is considered: Given a multivariable system with m inputs and r outputs and an $m \times r$ matrix whose nonnegative (i,j)'th element represents the cost of setting up a feedback link from the j'th output to the i'th input, find a set of feedback links with minimum total cost, which does not give rise to fixed modes. Utilizing the graph-theoretic characterization of structurally fixed modes, the problem is decomposed into two subproblems, which are then solved by using concepts and results from network theory. The combination of the optimum solutions of the subproblems provides a suboptimal solution to the original problem.

Keywords: Multivariable systems, feedback pattern, constrained control structure, digraph, structural representation of systems, structurally fixed modes.

ÖZET

ÇOKDEĞİŞKENLİ SİSTEMLERİN KISITLI DENETİM YAPISI ALTINDA KARARLILAŞTIRILMASI

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Elektrik ve Elektronik Mühendisliği Yüksek Lisans
Tez Yöneticisi: Prof. Dr. M. E. Sezer
Haziran 1988

Bu tezde, doğrusal, zamanla değişmeyen çok girişli-çok çıkışlı bir sistemde, özdeğerlerin istenildiği gibi seçilmesine olanak sağlayan en ucuz geribesleme yapısının belirlenmesi problemi ele alınmıştır. Bilindiği gibi, verilen herhangi bir geribesleme yapısı için, sistemin özdeğerlerinin bu yapıya uygun denetleyiciler kullanılarak istenildiği gibi seçilebilmesi için gerek ve yeter koşul, sistemin verilen geribesleme yapısı altında değişmez özdeğerlerinin bulunmamasıdır. Bu gerçekten hareketle, problem, değişmez özdeğerlere yol açmayan ve toplam maliyeti en düşük geribesleme yapısının bulunması biçiminde ele alınmıştır. Çizge kuramsal yöntemler kullanılarak, problem devre-akış problemlerine dönüştürülebilen iki alt-probleme indirgenmiş; bu alt-problemlerin optimal çözümlerinin uygun biçimde birleştirilmesiyle başlangıçtaki problem icin bir altoptimal çözüm elde edilmiştir.

Anahtar Kelimeler: Çokdeğişkenli dizgeler, geribesleme örüntüsü, kısıtlı denetim yapısı, yönlü çizge, yapısal değişmez özdeğerler.

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1. INTRODUCTION

Many of today's problems are consequences of present day technology and societal and environmental processes which are highly complex, 'large' in dimension, and stochastic by nature. The meaning of 'large scale' is a very subjective one and there is no universal definition of a large scale system. One approach to determine the notion of 'large scale' is that a system is considered large scale if it can be decomposed or partitioned into a number of interconnected subsystems or 'small scale' systems for either computational or practical reasons (Ho and Mitter, [1]). Another viewpoint is that a system is large scale when its dimensions are so large that conventional techniques of modeling, analysis, control, design, and computation fail to give reasonable solutions with reasonable computational efforts. In other words, a system is large when it requires more than one controller (Mahmoud, [2]).

The control procedures established in classical and modern control theory contexts, such as

1. modeling procedures which consists of differential equations, input output transfer functions, and state space formulations,

2. behavioral procedures of systems such as controllability, observability, and stability tests, and application of such criteria as Routh Hurwitz, Nyquist, etc.,

3. control procedures such as series compensation, pole placement, optimal control etc.

had all an underlying 'centrality' assumption (Sandell et al., [3]), i.e. all the calculations based upon system information and the information itself are localized at a given center, very often a geographical position.

An important characteristic of most large scale systems is that the centrality fails to hold due to either the lack of centralized computing capability or centralized information. The important points regarding large scale systems are that they often model real life systems dealing with society, business, management, the economy, the environment, energy, data networks, power networks, space structures, transportation, aerospace water resources, ecology, large scale integrated devices, and feasible flexible manufacturing networks, to name a few. The parts of these systems are often separated geographically, and their treatment requires consideration of not only economic costs, as is common in centralized systems, but also such important issues as reliability of communication links, value of information, etc.. Under these motivations two basic control strategies have been developed for large scale systems: 'hierarchical' and 'decentralized' control structures (Jamshidi [4]).

The underlying idea in decentralized control is to localize data gathering, storage, and control efforts at a number of control stations, which are often geographically separated. The input and output sets of the controllers are disjoint and each controller works independently of any others. Thus, decentralized control is a typical example of a constrained control structure, where local inputs are controlled by feeding back the information available at local measurement stations. The nonclassical control scheme arises naturally in many contexts, ranging from such engineering systems as multi-area power systems [5], [6] or data communication systems [7], [8], to economic systems involving several national agencies with regulatory power [9] or team decision and adjustment processes.

A systematic approach to decentralized control problem has been first developed by Wang and Davison [10], who introduced the fixed modes and related it to stabilizability of systems having a decentralized control structure. Later Corfmat and Morse [11], and Anderson and Clements [12] have achieved a more refined characterization of fixed modes and a deeper insight into the problem of decentralized control. Decentralized fixed modes of a linear system are the modes that cannot be shifted by constant decentralized output feedback. Hence they are related to both the internal structure (eigenstructure) of the system, and the feedback structure. Therefore, a characterization of fixed modes requires exact knowledge of systeparameters. Besides high dimensionalities, nonlinearities and complexities of interconnection in large scale systems cause computational difficulties in modeling, control and optimization as well as in the fundamental issues of stability, controllability, and observability. To avoid the difficulties due to lack of exact knowledge of system parameters, it may be preferrable to approach a large scale system from a structural point of view.

In the structural approach, only the qualitative properties are considered. In these analyses graph theory is used as one of the most powerful tools for representation of the structure of complex systems and for qualitative investigation of their properties. Graphical formulation aimes at obtaining results which are free of parameter values. Since all computations involve only binary variables, it is extremely easy to obtain results concerning the structure of the system. In addition, the same results are valid for all particular systems which are in the same structural equivalence class. Of course, the results obtained through graphical analysis do not explain everything about the system, but it is always advisable to make a structural analysis on the system in order to have preliminary knowledge for potential properties before going into more complex algebraic computations involving actual system parameters.

One of the earliest applications of graph-theoretic approach to control problems is the analysis of controllability properties of a linear system formulated by Lin [13]. Observing that the loss of controllability in a linear system may result from either a perfect matching of system parameters or due to the lack of sufficient interaction among system variables, Lin has developed the concept of structural controllability to characterize single-input systems which are either controllable or can be made controllable by changing the strength of interaction among certain variables. Thus structural controllability is a property of a class of systems having the same structure rather than a particular member of this class. As we shall explain in Chapter 2, the structure of a system can be conveniently described by a directed graph which allows for an extremely simple characterization of structural controllability. The concept and characterization of structural controllability have later been extended by Shields and Pearson [14], and Glover and Silverman [15] to multi-input systems.

As in the case of an uncontrollable or unobservable mode, a decentralized fixed mode may originate from two distinct sources: It is either a consequence of a perfect matching of system parameters (in which case, a slight change of the parameters can eliminate the mode), or it is due to a special structure of the system (in which case, the mode remains fixed no matter how much the parameters are perturbed, as long as the original structure is preserved).

From a physical point of view, only the latter type of fixed modes are important, not only because it is very unlikely to have an exact matching of the parameters, but also because it is not possible to know whether such a matching occurs in a given system. Motivated by this paralellism between uncontrollable and fixed modes, Sezer and Siljak [16] gave a structural interpretation of decentralized fixed modes by extending the ideas and results developed by Lin, and provided an algebraic characterization of structurally fixed modes. Based on their characterization, Pichai and colleagues [17] and Reinschke [18] independently developed graphical criteria for testing existence of structurally fixed modes. Their results also revealed that decentralization constraint was not essential for the problem formulation; that is, both the notion of fixed modes and the corresponding stabilizability condition could be extended to arbitrary feedback structure constraints.

Although the graphical characterization of structurally fixed modes (which we will discuss in the next chapter) is very convenient for analysis purposes, it is not readily applicable to the inverse problem of identifying the best feedback pattern among the feasible ones (those that avoid fixed modes). The only known attempt in this direction was made by Sezer [19], who considered the problem of constructing a feasible feedback pattern which includes a minimum number of feedback links. However, only partial results were obtained. The objective of this work is to formulate and suggest a solution method for a more general optimization problem involving the minimization of the total cost of the feedback links in a feasible pattern. More precisely we consider the following problem.

"Given a multivariable system with m inputs and r outputs, and an m$\times$r matrix K^c whose nonnegative element k^c_{ij} represents the cost of the feedback from the j-th output to the i-th input, find a feasible set of feedback links with minimum total cost."

Once a solution to the above problem is obtained, a set of dynamic feedback compensators compatible with the optimal pattern can be designed using known techniques (e.g. following the Wang-Davison approach). It should be emphasized that, in the formulation above, no cost is associated with the compensators themselves, but rather with the information channels.

Obviously, a brute force approach to the above problem via a direct search of all possible feedback patterns is computationally intractable as there are $2^{m\times r} - 1$ nontrivial patterns. In this thesis, we propose a computationally tractable procedure to construct a suboptimal feedback pattern through a sequential optimization process. To avoid the difficulties involved in algebraic tests for fixed modes, we formulate the problem in a structural framework, where we consider an equivalence class of systems having the same structure rather than a particular system with fixed parameters. This way we reduce the problem to one of binary nature, which can be attacked by powerful graph-theoretic techniques.

As is known, there are two types of structurally fixed modes, both resulting from lack of sufficient interconnections among the system variables. We show that the problem of choosing an optimal set of feedback links that avoid either of these two types of structurally fixed modes can be reformulated as a network flow problem (e.g. Bazaraa and Jarvis [20], Chaera, Moore and Ghare [21]). In other words, we decompose the problem into two network flow problems, each of which is associated with one type structurally fixed modes. Unfortunately, the two network flow problems are quite different in nature, and therefore, cannot be combined into a single problem which can be solved without total enumeration. This necessitates to employ a sequential optimization procedure, which consists of solution of one of the problems, modification of the feedback costs, solution of the other problem with modified costs, and combination of the two solutions. This way, a suboptimal solution to the overall problem is obtained.

1.1 Outline of the thesis

In Chapter 2 we introduce the graph theoretic terminology, review some basic concepts and results from graph theory and discuss structural representation of linear systems by system digraphs and system structure matrices. In Chapter 3 we deal with the problem of controlling a linear multivariable system by structurally constrained dynamic feedback. Particularly, fixed modes, structurally fixed modes and their algebraic and graph theoretic characterizations are considered. In Chapter 4, we formulate the problem of optimum feedback pattern selection, and decompose it into two subproblems. We show that these subroblems can be reformulated as network flow problems. For each problem we provide solution algorithms. These algorithms give a suboptimal solution to the overall problem, which is shown to be %100 more costly than the optimal one in the worst case. Finally, Section 5 contains conclusions and further comments on the formulation of the problem.

1.2 Notation

Throughout the thesis, matrices and vectors are denoted by upper and lowercase italic letters, binary matrices by uppercase boldface letters, and abstract objects such as a set, a system etc. by script letters. A superscript c indicates the cost associated with a matrix or its elements.

2. STRUCTURAL REPRESENTATION OF SYSTEMS

In this chapter we introduce the graph theoretic terminology, review some basic concepts and results from graph theory, and discuss structural representation of linear systems by system digraphs and system structure matrices. The material on graph theory is borrowed mainly from Norman, Harary and Cartwright [22] and from Deo [23], with minor changes or additions. A detailed treatment of structural representation of systems by digraphs can be found in Coates [24] and Siljak [25]. The concept of structural equivalence of systems was introduced by Shields and Pearson [14], who also considered the problem of computing generic rank of a matrix. The generic rank problem has also been considered by Duff [26] and Papadimitriou and Tsitsiklis [27] in different contexts.

2.1 Digraphs

A digraph $\mathcal{D} = (\mathcal{V}, \mathcal{E})$ is an ordered pair, where $\mathcal{V}$ is a finite set of vertices, and $\mathcal{E}$ is a finite set of ordered pairs of vertices called edges. If $(v_j, v_i) \in \mathcal{E}$, then vertex v_j is said to be adjacent to vertex v_i. Adjacency relation defines a binary matrix $\mathbf{M} = (\mathbf{m}_{ij})$, called the adjacency matrix of $\mathcal{D}$, such that $\mathbf{m}_{ij} = 1$ if and only if $(v_j, v_i) \in \mathcal{E}$. A digraph is completely characterized by its adjacency matrix. A sequence of edges $((v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k))$ where all vertices are distinct, is called a path from v_1 to v_k. In this case v_k is said to be reachable from v_i. Reachability relation too can be described by a binary matrix $\mathbf{R} = (\mathbf{r}_{ij})$ such that $\mathbf{r}_{ij} = 1$ if and only if v_j reaches v_i. The reachability matrix $\mathbf{R}$ is related to the adjacency matrix $\mathbf{M}$ as explained below:

Note that $\mathbf{M}$ can be considered as one step reachability matrix. In other words, for a fixed v_j, the set of vertices that v_i can be reached from v_j in one step are exactly those for which $\mathbf{m}_{ij}$=1. Now defining all multiplications and additions as Boolean operations, $\mathbf{M}^2 = \mathbf{M} \times \mathbf{M}$ characterizes two step reachability of $\mathcal{D}$. Similarly $\mathbf{M}^k = \mathbf{M}^{k-1} \times \mathbf{M}$ shows all the nodes reachable from a node in k steps. Thus, $\mathbf{R} = \mathbf{I} + \mathbf{M} + \mathbf{M}^2 + ...$. Noting that, if a vertex is reachable from another one at all, then it is reachable in at most n-1 steps, the infinite series above can be truncated at k=n-1 resulting in $\mathbf{R} = \mathbf{I} + \mathbf{M} + \mathbf{M}^2 + ... + \mathbf{M}^{n-1}$.

A subgraph of $\mathcal{D}$ is a digraph $\mathcal{D}_s = (\mathcal{V}_s, \mathcal{E}_s)$, where $\mathcal{V}_s \subset \mathcal{V}$ and $\mathcal{E}_s$=$\{(v_j, v_i) \in \mathcal{E} : \ v_j, v_i \in \mathcal{V}_s\}$. Let us define a connectedness relation on $\mathcal{D} = (\mathcal{V}, \mathcal{S})$ as

i) Adjacent nodes are connected

ii) Any two nodes connected seperately to a third one are connected.

We see that the relation is an equivalence relation which defines several equivalence classes on $\mathcal{V}$ namely $\mathcal{V}_1, ..., \mathcal{V}_s$. The subgraphs $\mathcal{D}_i = (\mathcal{V}_i, \mathcal{E}_i)$, i=1,...,s, are the connected components of $\mathcal{D}$. It is seen that $\mathcal{E}_i \cap \mathcal{E}_j = \emptyset$ for $i \neq j$, and $\cup \mathcal{E}_i = \mathcal{E}$.

Another equivalence relation of $\mathcal{V}$ is strong connectedness, which takes into account the direction of edges: Two vertices are said to be strongly connected if they are mutually reachable. A maximal subgraph containing strongly connected vertices is called a strong component of $\mathcal{D}$. Strong components are uniquely determined. The set of edges of the strong components also satisfy that $\mathcal{E}_i \cap \mathcal{E}_j = \emptyset$ for $i \neq j$ but in general $\cup \mathcal{E}_i \neq \mathcal{E}$.

A sequence of edges $((v_1, v_2), (v_2, v_3), ..., (v_{k-1}, v_k), (v_k, v_1))$ where all vertices are distinct, is called a cycle in $\mathcal{D}$. If $\mathcal{D}$ contains no cycles it is said to be acyclic. In an acyclic digraph, each vertex is a strong component by itself. If $\mathcal{D}$ is an acyclic digraph, then by a level assignment algorithm [22] its vertices can be enumerated such that $(v_j, v_i) \in \mathcal{E}$ only if $j < i$. That is, there exists a permutation matrix $\mathbf{P}$ such that $\mathbf{P}^T\mathbf{M}\mathbf{P}$ is lower triangular.

If we partition the vertex set $\mathcal{V}$ arbitrarily into subsets $\{ \mathcal{V}_i \}$, then the condensation of $\mathcal{D}$ with respect to this partition is the digraph $\mathcal{D}^* = (\mathcal{V}^*, \mathcal{E}^*)$, where $\mathcal{V}^*$ contains a vertex v_i^* corresponding to each subset $\mathcal{V}_i$, and $(v_j^*, v_i^*) \in \mathcal{E}^*$ if and only if a vertex in $\mathcal{V}_j$ reaches a vertex in $\mathcal{V}_i$. Note that if $\mathcal{V}$ is partitioned into connected components then the corresponding condensation consists of isolated vertices, and if it is partitioned into strong components, then the condensation $\mathcal{D}^*$ is acyclic. In this case, since the adjacency matrix of $\mathcal{D}^*$ can be permuted to a lower triangular form, it is clear that the adjacency matrix of $\mathcal{D}$ can be permuted to a block triangular form, where the diagonal blocks are the adjacency matrices of the corresponding strong components.

2.2 System Structure Matrices and System Digraphs

A structured matrix (or sometimes structure matrix) is a matrix with fixed zeros and arbitrary unrelated indeterminates in the remaining locations. Let $\mathbf{M} = (\mathbf{m}_{ij})$ be a $p \times q$ structured matrix with ν nonzero parameters in specified locations. Then $\mathbf{M}$ can be viewed as a transformation from $\mathcal{R}^{\nu}$ into $\mathcal{R}^{p\times q}$, such that for each distinct $d \in \mathcal{R}^{\nu}$, $\mathbf{M}(d)$ is a unique fixed matrix in $\mathcal{R}^{p\times q}$. It is easy to see that $\mathbf{M}$ defines an equivalence class in $\mathcal{R}^{p\times q}$ of structurally equivalent matrices: We say that two matrices $M_1,\ M_2 \in \mathcal{R}^{p\times q}$ are structurally equivalent if there is a one-to-one correspondence between the locations of their nonzero entries. That is M_1 and M_2 are structurally equivalent if $M_1 = \mathbf{M}(d_1)$ and $M_2 = \mathbf{M}(d_2)$ for some structure matrix $\mathbf{M}$ and $d_1,\ d_2 \in \mathcal{R}^{\nu}_{\phi}$, where $\mathcal{R}^{\nu}_{\phi} = \{d = [d_1\ d_2\ ...d_\nu]^T\ :\ d_i \neq 0,\ i = 1,2,...\nu\}$. For convenience, we visualize a structured matrix $\mathbf{M}$ as a binary matrix, in which the indeterminate elements are represented by 1.

Let M be a $p \times q$ matrix. The generic rank of M, denoted by g.r.(M), is the maximum of the rank of the all matrices that are structurally equivalent to M. In other words if we denote the structure matrix of M by $\mathbf{M}$, where the number of indeterminates is ν, then

$$g.r.(M) = \ max_{d\in\mathcal{R}^{\nu}}\{\ rank[\mathbf{M}(d)]\}.$$

it is easy to see that g.r.$(M) = r$ for some $r \leq \min(p,q)$ if and only if there exist at most r nonzero elements of M at different rows and columns. g.r.$(M) < r$ for some $r \leq \min(p,q)$ if and only if M contains a zero submatrix with the sum of the number of rows and number of columns no less than $p + q + 1 - r$ [14]. In Appendix A we provide an algorithm which computes the generic rank of a matrix. It can be shown that the set $D = \{d \in \mathcal{R}^{\nu} : rank[\mathbf{M}(d)] < g.r.(M)\}$ is a variety in $\mathcal{R}^{\nu}$. In other words if $d_0 \in D$, then in each neighbourhood of d_0 there exists elements of $\mathcal{R}^{\nu}$ different than those of D.

Consider a linear, multivariable system described as

$$\begin{aligned} \mathcal{S}:\ \ \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

where $x \in \mathcal{R}^n$, $u \in \mathcal{R}^m$ and $y \in \mathcal{R}^r$ denote the states, inputs and outputs of $\mathcal{S}$ respectively and A, B and C are constant matrices of appropriate dimensions. We denote $\mathcal{S}$ by the triple (A, B, C).

Associated with $\mathcal{S}$ we construct a binary matrix.

$$\mathbf{S} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m & \mathbf{0} \\ \mathbf{C} & \mathbf{0} & \mathbf{I}_r \end{bmatrix}, \tag{2}$$

called the system structure matrix, where $\mathbf{A} = (\mathbf{a}_{ij})$ with $\mathbf{a}_{ij}$ =1 if and only if $a_{ij} \neq 0$ and $\mathbf{B}$ and $\mathbf{C}$ are defined similarly.

The digraph $\mathcal{D} = (\mathcal{V}, \mathcal{E})$, which assumes $\mathbf{S}$ as its adjacency matrix is called the system digraph of $\mathcal{S}$. Due to the structure of $\mathcal{S}$, the vertices of the system digraph $\mathcal{D}$ are associated with the state, input and output variables of $\mathcal{S}$. That is, $\mathcal{V} = \mathcal{X} \cup \mathcal{U} \cup \mathcal{Y}$, where $\mathcal{X} = \{x_1, x_2, ..., x_n\}$, $\mathcal{U} = \{u_1, u_2, ..., u_m\}$, and $\mathcal{Y} = \{y_1, y_2, ..., y_r\}$ are the sets of state, input and output vertices. $\mathcal{E}$ consists of edges of the form (x_j, x_i), (u_j, x_i), (x_j, y_i), and self loops at the input and output vertices. $(x_j, x_i) \in \mathcal{E}$ for some i, j, if and only if $a_{ij} \neq 0$, that is the variable x_j occurs in the equation for $\dot{x}_i$. Similarly, $(u_j, x_i) \in \mathcal{E}$ if and only if $b_{ij} \neq 0$, and $(x_j, y_i) \in \mathcal{E}$ if and only if $c_{ij} \neq 0$. Thus, $\mathcal{D}$ completely specifies the structure of the system $\mathcal{S}$. The subgraph $\mathcal{D}_{xu} = (\mathcal{X} \cup \mathcal{U}, \mathcal{E}_{xu})$ of $\mathcal{D}$ obtained by removing output verices and the edges connected to them, is called the output-truncated subgraph. $\mathcal{D}_{xu}$, which has the adjacency matrix

$$\mathbf{S}_{xu} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix},$$

is suitable in characterizing the structural properties of the pair (A,B). The input-truncated subgraph $\mathcal{D}_{xy} = (\mathcal{X} \cup \mathcal{Y}, \mathcal{E}_{xy})$, and the input/output-truncated subgraph $\mathcal{D}_x = (\mathcal{X}, \mathcal{E}_x)$ are defined similarly.

Two systems which have the same system digraph are called structurally equivalent. If $\mathcal{S} = (A, B, C)$ and $\tilde{\mathcal{S}} = (\tilde{A}, \tilde{B}, \tilde{C})$ are structurally equivalent systems, then their system structure matrices are related as

$$\begin{bmatrix} \tilde{\mathbf{A}} & \tilde{\mathbf{B}} & 0 \\ 0 & \mathbf{I} & 0 \\ \tilde{\mathbf{C}} & 0 & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_x^T\mathbf{A}\mathbf{P}_x & \mathbf{P}_x^T\mathbf{B}\mathbf{P}_u & 0 \\ 0 & \mathbf{I} & 0 \\ \mathbf{P}_y\mathbf{C}\mathbf{P}_x & 0 & \mathbf{I} \end{bmatrix},$$

where $\mathbf{P}_x$, $\mathbf{P}_u$ and $\mathbf{P}_y$ are permutation matrices representing a reordering of the state, input and output vertices of $\mathcal{D}$.

Structurally equivalent systems form an equivalence class, which can conveniently be represented by the system digraph $\mathcal{D}$. A property is said to be a structural property of a system if it holds for at least one member of the equivalence class to which that particular system belongs.

Let $\mathbf{K} = (\mathbf{k}_{ij})$ be an m $\times$ r binary matrix, which specifies a feedback pattern for the system $\mathcal{S}$ in (1) such that $\mathbf{k}_{ij}$=1 if and only if a feedback from output y_j to input u_i is allowed. Let K be a constant matrix in the equivalence class $\mathbf{K}$. Then, the closed-loop system consisting of $\mathcal{S}$ and the feedback law

$$\mathcal{F} : u = Ky + v,$$

where $v \in R^m$ stands for reference inputs, is represented by the closed-loop system structure matrix

$$\hat{\mathrm{S}} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m & \mathbf{K} \\ \mathbf{C} & \mathbf{0} & \mathbf{I}_r \end{bmatrix}. \tag{3}$$

The closed loop digraph of (1) is defined as $\hat{\mathcal{D}} = (\mathcal{V}, \mathcal{E} \cup \mathcal{E}_K)$ where $\mathcal{E}_K = \{(y_j, u_i) : \; k_{ij} = 1\}$ contains the feedback edges.

2.3 Structural Controllability and Observability

The concept of structural controllability (observability) has been introduced by Lin [13], who pointed out that a loss of controllability of a system occurs either due to parameter values or due to the system structure. In the former case, a slight perturbation of some of the nonzero parameters restore controllability, while in the latter case the system remains uncontrollable no matter how the nonzero parameters are chosen. Lin defined a system which is structurally equivalent to a controllable (observable) one to be structurally controllable (observable). Thus, structural controllability (observability) is a property of a class of structurally equivalent systems rather than a particular member of this class. It has been shown by Lin for single-input systems, and by Shields and Pearson [14] for multi-input systems, that a system $\mathcal{S} = (A, B, C)$ is structurally controllable if and only if

a) g.r. $\begin{bmatrix} A & B \end{bmatrix} = n$, where n is the order of A, and

b) the output truncated system digraph $\mathcal{D}_{xu}$ is input reachable, that is each state vertex is reachable by an input vertex.

Obviously, the dual conditions for structural observability are

á) g.r. $\begin{bmatrix} A \\ C \end{bmatrix} = n$

$\acute{b}$) input truncated system digraph $\mathcal{D}_{xy}$ is output reachable, that is, each state vertex reaches an output vertex.

Since structurally equivalent systems are characterized by the same system digraph, it is natural that the conditions for structural controllability (observability) can be expressed in terms of the system digraph. This graph theoretic characterization of structural controllability, which was provided by Lin for single-input systems, can be stated as follows.

"A system $\mathcal{S}$ is structurally controllable (observable) if and only if its output (input) truncated digraph is spanned by a union of input (output) cacti",

where a cactus is a special digraph such that removal of any edge violates either or both conditions (a) and (b).

3. CONTROL UNDER INFORMATION STRUCTURE CONSTRAINTS

In this chapter, we consider the problem of controlling a linear multivariable system using structurally constrained dynamic feedback. In particular, we focus our attention to fixed modes and their algebraic and graph-theoretic characterization.

3.1 Fixed Modes Under Constrained Feedback

Consider the multivariable system $\mathcal{S}$ described in (1), together with a given feedback pattern specified by some binary matrix $\mathbf{K}$. Corresponding to each input u_i of $\mathcal{S}$, we define an index set

$$\mathcal{J}_i = \{j : \mathbf{k}_{ij} = 1\}, \quad i = 1, 2, ..., m \tag{1}$$

which specifies those outputs y_j from which feedback to input u_i is allowed. Accordingly, permissible dynamic feedback controllers are described by

$$\mathcal{C}_i : \quad \begin{array}{l} \dot{z}_i = F_i z_i + \sum_{j \in \mathcal{J}_i} q_{ij} y_{ij} \\ u_i = h_i^T z_i + \sum_{j \in \mathcal{J}_i} k_{ij} y_j \end{array} , \tag{2}$$

where $z_i \in \mathcal{R}^{l_i}$ is the state of the i'th controller $\mathcal{C}_i$, and F_i, g_{ij}, h_i and k_{ij} are constant matrices, vectors and scalars of appropriate dimension.

We note that, since no special structure is imposed on the feedback pattern matrix $\mathbf{K}$, the set of controllers in (2) represent the most general form of a constrained compensation scheme. For example it includes decentralized control as a special case, where $\mathbf{K}$ is a block diagonal.

Generalizing the definition of decentralized fixed modes by Wang and Davison, we define

$$\Lambda_{\mathbf{K}} = \bigcap_{K \in \mathbf{K}} \Lambda(A + BKC) \tag{3}$$

to be the set of fixed modes of $\mathcal{S}$ with respect to $\mathbf{K}$, where $\Lambda(.)$ denotes the set of eigenvalues of the indicated matrix. The role of fixed modes in stabilizability of $\mathcal{S}$ is clarified by the following result of Sezer and Siljak [16], which is a generalization of the corresponding result of Wang and Davison.

Theorem 3.1: The system $\mathcal{S}$ can be stabilized by using dynamic output feedback compensators of the form (2) if and only if the set of fixed modes $\Lambda_{\mathbf{K}}$ contains only the elements with negative real parts.

The following result by Pichai and colleagues [17] is a generalization of the algebraic characterization of decentralized fixed modes by Anderson and Clements [12]:

Theorem 3.2: A complex number σ is a fixed mode of the system $\mathcal{S}$ with respect to $\mathbf{K}$ if and only if there exists a subset $\mathcal{I}$ of the set $\mathcal{M} = \{1, 2, ..., m\}$ such that

$$rank \begin{bmatrix} A - \sigma I & B_I \\ C_J & 0 \end{bmatrix} < n. \tag{4}$$

where $\mathcal{J} = \cup_{i \in M - I} \mathcal{J}_i$.

As we mentioned earlier fixed mode of a system originates from two sources. It is either a result of the perfect matching of the system parameters or it is due to the structure of the triple (A, B, C). We illustrate these by an example.

Example 3.1

Consider a system $\mathcal{S}$ with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{5}$$

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

and a feedback pattern $\mathbf{K} = \mathbf{I}$. For any

$$K = diag\{k_1, k_2\} \in \mathbf{K}, \tag{6}$$

the closed loop matrix

$$A + BKC = \begin{bmatrix} 0 & 1 & k_1 \\ 1 & 1 & 0 \\ k_2 & 0 & 1 \end{bmatrix}$$

has an eigenvalue at $\sigma = 1$ independent of k_1 and k_2, which is by definition a fixed mode of $\mathcal{S}$ with respect to $\mathbf{K}$. However, if we slightly perturb the nonzero element in (3,3) position of A to obtain

$$A_\varepsilon = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1+\varepsilon \end{bmatrix} \tag{7}$$

where ε is an arbitrarily small but a nonzero number, it is easy to see that the resulting system $\mathcal{S}^\epsilon$ would have no fixed modes with respect to $\mathbf{K}$. The fixed mode of $\mathcal{S}$ is caused by a perfect matching of the nonzero elements of A.

If, on the other hand, the matrix A in (5) were

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{8}$$

then the system $\mathcal{S}$ would have a fixed mode at $\sigma = 0$ no matter how the nonzero elements of the triple (A, B, C) were perturbed. In this case, the fixed mode is caused by the structure of the system and it is independent of the values taken by the nonzero elements of (A, B, C). This motivates the need for a structural interpretation of fixed modes, which we consider in the next section.

3.2 Structurally Fixed Modes

Recall that two systems are structurally equivalent if their system digraphs are the same. A system $\mathcal{S}$ is said to have structurally fixed modes with respect to a feedback structure constraint $\mathbf{K}$ if every system which is structurally equivalent to $\mathcal{S}$ has fixed modes with respect to $\mathbf{K}$. If $\mathcal{S}$ has no structurally fixed modes, we denote this fact symbolically by $\Lambda_{\mathbf{K}} = \emptyset$

An algebraic characterization of structurally fixed modes was given by Sezer and Siljak [16]:

A) there exists an $\mathcal{I} \subset \mathcal{M}$ such that

$$g.r. \begin{bmatrix} A & B_I \\ C_J & 0 \end{bmatrix} < n,$$

B) there exists an $\mathcal{I} \subset \mathcal{M}$ and a permutation matrix P such that

$$P^T A P = \begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \quad P^T B_I = \begin{bmatrix} 0 \\ 0 \\ B_3^I \end{bmatrix},$$

$$C_J P = \begin{bmatrix} C_1^J & 0 & 0 \end{bmatrix}.$$

where $\mathcal{M}$ and $\mathcal{J}$ are as defined in Theorem 3.2.

As an illustration of Theorem 3.3 consider $\mathcal{S} = (A, B, C)$ with A given in (8) and B and C given in (5). For **K**=**I**, if we let $\mathcal{I}$={1} and $\mathcal{J} = \mathcal{J}_2 = \{2\}$ then

$$g.r. \begin{bmatrix} A & B_I \\ C_J & 0 \end{bmatrix} = g.r. \left[\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \end{array}\right] = 2 < n.$$

that is, condition (ii) of Theorem 3.3 is violated. Consequently, $\mathcal{S}$ has a structurally fixed mode at $\sigma = 0$ with respect to **K**.

Since structurally fixed modes are a property of an equivalence class of systems, it is natural to expect that they can be characterized in terms of the system digraph. This is indeed the case as stated by the following theorem by Pichai and colleagues [17].

Theorem 3.4: A structured system $\mathcal{S}$ has no structurally fixed modes with respect to a feedback pattern **K** if and only if the following two conditions are satisfied:

A. The closed-loop system digraph $\hat{\mathcal{D}}$ is covered by a collection of vertex disjoint cycles.

B. Each state vertex of $\hat{\mathcal{D}}$ occurs in a strong component that contains a feedback edge.

It should be noted that the graphical conditions (A) and (B) of Theorem 3.4 are negations of the algebraic conditions (A) and (B) of Theorem 3.3. To illustrate this fact, consider the closed-loop system digraph $\hat{\mathcal{D}}$ shown in Fig.

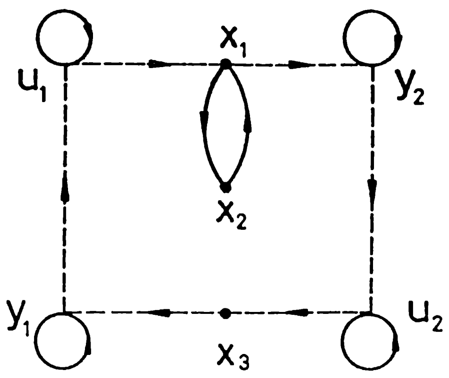


Figure 3.1: Closed loop digraph of the system in Example 3.1

illustrate this fact, consider the closed-loop system digraph $\hat{\mathcal{D}}$ shown in Fig. 3.1, corresponding to the system $\mathcal{S}$ considered previously, where the feedback edges are indicated by broken lines. From the figure it is observed that although all state vertices occur in the same strong component of $\hat{\mathcal{D}}$ which includes both feedback edges, one cannot find a family of disjoint cycles that cover all the state vertices. Hence, condition (A) of Theorem 3.4 is violated, and consequently $\mathcal{S}$ has a structurally fixed mode at $\sigma = 0$.

We also note that condition (A) of Theorem 3.4 is equivalent to the closed-loop system structure matrix $\mathbf{S}(\mathbf{K})$ having full generic rank. Therefore, an equivalent statement of Theorem 3.4 can be given as follows:

Theorem 3.5: A structured system $\mathcal{S}$ has no structurally fixed modes with respect to a feedback pattern $\mathbf{K}$ if and only if

A)

$$g.r. \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m & \mathbf{K} \\ \mathbf{C} & \mathbf{0} & \mathbf{I}_r \end{bmatrix} = n + m + r,$$

B) No strong component of $\hat{\mathcal{D}}$ contains only state vertices.

Let us now return to Theorem 3.4. Suppose that condition (B) fails to hold, and let $\mathcal{D}_S = (\mathcal{X}_S, \mathcal{E}_S)$ be the subgraph of $\hat{\mathcal{D}}$ which includes all the state vertices that violate the condition. Then, $\mathcal{D}_S$ corresponds to a principal submatrix of $\mathbf{A}$ in S. The modes of the associated structured subsystem can conveniently be termed as B_type structurally fixed modes. Now, consider the subgraph which is obtained from $\hat{\mathcal{D}}$ by removing all the vertices in $\mathcal{X}_S$ and the edges connected to them. Clearly, this subgraph satisfies condition (B). Let k be the number of state vertices in this subgraph which are left k more structurally fixed modes which we term as A_type structurally fixed modes. Note that A_type structurally fixed modes are always at the origin, and cannot be associated with a part of $\mathcal{S}$. This way, structurally fixed modes of a system can be classified into two distinct groups, both of which result from insufficient interconnection among system variables. This classification of fixed modes is useful in decomposing the optimization problem, which we consider next.

4. OPTIMAL FEASIBLE FEEDBACK STRUCTURES

In this chapter , we formulate the optimum feedback pattern selection problem, and decompose it into two subproblems. We show that these subproblems can be formulated as network flow problems. We also conclude that each problem has a different nature in the sense of solution complexities. For each problem we provide solution algorithms. These algorithms give a suboptimal solution to the overall problem, which is shown to be %100 more costly than the optimal one in the worst case.

4.1 Problem Statement and Decomposition

The problem we consider is pole placement in a system $\mathcal{S}$ using minimum cost dynamic feedback compensators. For this purpose we define the total cost of a given feedback pattern $\mathbf{K}$ to be

$$c(\mathrm{K}) = \sum_{\mathbf{k}_{ij}=1} k^c_{ij}. \tag{1}$$

and formulate our problem as

$$\mathcal{P}: \quad \left.\begin{array}{l} min \;\; c(\mathrm{K}) \\ s.t. \;\; \Lambda_{\mathrm{K}} = \emptyset \end{array}\right\}. \tag{2}$$

In (1), k^c_{ij} denotes the cost of setting up a feedback link from output y_j to input u_i. If a particular feedback link (y_j, u_i) is not to be used at all, then this constraint is represented by letting $k^c_{ij} = \gamma$, where γ is a very large positive number. It should be noted that, in the problem formulation we restrict our attention only to structurally fixed modes, which allows us to characterize the feasible feedback patterns in terms of the closed-loop system digraph $\hat{\mathcal{D}}$. Still, however, the feasibility condition $\Lambda_{\mathrm{K}} = \emptyset$ involves two tests for A_ and B_type structurally fixed modes, which cannot be combined into a single graphical condition. Clearly, the only way to solve problem $\mathcal{P}$ is to employ a clever enumeration technique if not total enumeration.

To avoid the computational burden of total enumeration, we propose to decompose the problem $\mathcal{P}$ into two simpler problems involving only A_ and B_types of structurally fixed modes :

$$\mathcal{P}_A : \quad \left.\begin{array}{l} min\ \ c(\mathbf{K}) \\ s.t.\ \ \Lambda^A_{\mathbf{K}} = \emptyset \end{array}\right\} \tag{3}$$

and

$$\mathcal{P}_B : \quad \left.\begin{array}{l} min\ \ c(\mathbf{K}) \\ s.t.\ \ \Lambda^B_{\mathbf{K}} = \emptyset \end{array}\right\} \tag{4}$$

where $\Lambda^A_{\mathbf{K}}$ and $\Lambda^B_{\mathbf{K}}$ refer to the corresponding types of fixed modes. If $\mathbf{K}^o_A$ and $\mathbf{K}^o_B$ are optimal solutions of problems $\mathcal{P}_A$ and $\mathcal{P}_B$, then

$$\mathbf{K}^s = \mathbf{K}^o_A + \mathbf{K}^o_B, \tag{5}$$

where $(+)$ denotes Boolean OR operation, is a feasible feedback pattern for problem $\mathcal{P}$ such that

$$max\ \{c(\mathbf{K}^o_A), c(\mathbf{K}^o_B)\} \leq c(\mathbf{K}^o) \leq c(\mathbf{K}^s) \leq c(\mathbf{K}^o_A) + c(\mathbf{K}^o_B), \tag{6}$$

where $\mathbf{K}^o$ is the original solution of the original problem $\mathcal{P}$. From (6) it follows that

$$c(\mathbf{K}^s) \leq 2c(\mathbf{K}^o), \tag{7}$$

that is, the solution $\mathbf{K}^s$ obtained through a decomposition of $\mathcal{P}$ is at most 100% more costly than the optimum solution. We can, therefore, think of $\mathbf{K}^s$ to be a suboptimal solution of problem $\mathcal{P}$.

We note that once an optimal solution to one of the above problems is obtained, then some of the feedback links that appear in the corresponding feedback pattern may help satisfy the feasibility condition of the other problem without any additional cost. This suggests a sequential optimization procedure, where the two problems are solved sequentially with the cost matrix modified after solving the first problem. Thus, with $\mathbf{K}^o_A=(\mathbf{k}^{oA}_{ij})$ being a solution of problem $\mathcal{P}_A$, we modify the cost matrix K^c into $K^c_A=(k^{cA}_{ij})$, where

$$k_{ij}^{cA} = \begin{cases} 0 & , \ \mathbf{k}_{ij}^{oA} = 1 \\ k_{ij}^{c} & , \ otherwise \end{cases} , \tag{8}$$

and replace problem $\mathcal{P}_B$ with

$$\acute{\mathcal{P}}_B : \quad \left. \begin{array}{l} min \ c^A(\mathbf{K}) \\ s.t. \ \Lambda_{\mathbf{K}}^B = \emptyset \end{array} \right\} \tag{9}$$

where

$$c^A(\mathbf{K}) = \sum_{\mathbf{k}_{ij}=1} k_{ij}^{cA}. \tag{10}$$

Now, with $\acute{\mathbf{K}}_B^o$ being the optimum solution of problem $\acute{\mathcal{P}}_B$, we have

$$c(\acute{\mathbf{K}}_B^o) \leq c(\mathbf{K}_B^o). \tag{11}$$

Thus, defining the suboptimal solution obtained through the sequential optimization of the problems $\mathcal{P}_A$ and $\acute{\mathcal{P}}_B$ as

$$\mathbf{K}_{AB}^s = \mathbf{K}_A^o + \acute{\mathbf{K}}_B^o, \tag{12}$$

we have

$$c(\mathbf{K}_{AB}^s) \leq c(\mathbf{K}^s), \tag{13}$$

which shows that the loss due to decomposition is decreased by employing sequential optimization scheme.

Obviously, one could interchange the order of the two problems, and start with $\mathcal{P}_B$ instead. In this case, problem $\mathcal{P}_A$ would be replaced by

$$\acute{\mathcal{P}}_A : \quad \left. \begin{array}{l} min \ c^B(\mathbf{K}) \\ s.t. \ \Lambda_{\mathbf{K}}^A = \emptyset \end{array} \right\} \tag{14}$$

where K_B^c and $c^B(\mathbf{K})$ are similar to K_A^c and $c^A(\mathbf{K})$, and are defined after a solution $\mathbf{K}_B^o$ to problem $\mathcal{P}_B$ is obtained. In what follows, we drop the prime notation for convenience, with the understanding that $\mathcal{P}_B$ denotes the modified problem $\acute{\mathcal{P}}_B$ in the sequence $(\mathcal{P}_A, \acute{\mathcal{P}}_B)$ and $\mathcal{P}_A$ denotes $\acute{\mathcal{P}}_A$ in the sequence $(\mathcal{P}_B, \acute{\mathcal{P}}_A)$. We also employ the notation $\mathcal{P}_A(\mathcal{D}, K^c)$ or $\mathcal{P}_B(\mathcal{D}, K^c)$ to indicate explicitly the digraph and the cost matrix upon which $\mathcal{P}_A$ or $\mathcal{P}_B$ is formulated.

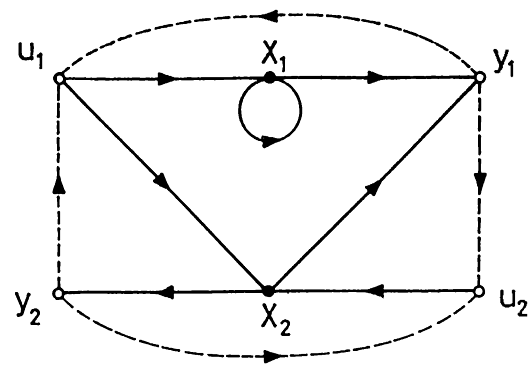


Figure 4.1: Digraph of the system in Example 4.1

Before considering solution procedures for $\mathcal{P}_A$ and $\mathcal{P}_B$, we would like to point out that, in general, the ultimate suboptimal solution depends on the order in which the two subproblems are solved as we demonstrate by an example below.

Example 4.1

Consider a structured system described by

$$\mathbf{S} = \left[\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]. \tag{15}$$

Let the cost matrix be given as

$$K^c = \begin{bmatrix} 4 & 2 \\ 3 & 5 \end{bmatrix}. \tag{16}$$

From the closed-loop system digraph $\hat{\mathcal{D}}$ shown in Fig. 4.1 the feasible feedback patterns for problem $\mathcal{P}$ can easily be identified as

$$\begin{bmatrix} 1 & \star \\ \star & \star \end{bmatrix} and \begin{bmatrix} \star & 1 \\ 1 & \star \end{bmatrix}, \tag{17}$$

where $\star$ denotes either a 0 or 1. The unique optimal solution of problem $\mathcal{P}$ can be obtained by inspection as

$$\mathbf{K}^o = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{18}$$

resulting into the optimal cost c($\mathbf{K}^o$)=4.

Now suppose problem $\mathcal{P}$ is decomposed into $\mathcal{P}_A$ and $\mathcal{P}_B$ which are solved sequentially with $\mathcal{P}_A$ first. Feasible patterns for $\mathcal{P}_A$ (not distinct) are

$$\begin{bmatrix} 1 & \star \\ \star & \star \end{bmatrix}, \begin{bmatrix} \star & 1 \\ \star & \star \end{bmatrix}, \begin{bmatrix} \star & \star \\ 1 & \star \end{bmatrix} \quad and \quad \begin{bmatrix} \star & \star \\ \star & 1 \end{bmatrix}, \tag{19}$$

and the optimal one is

$$\mathbf{K}_A^o = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{20}$$

Modifying the cost matrix K^c accordingly into

$$K_A^c = \begin{bmatrix} 4 & 0 \\ 3 & 5 \end{bmatrix}, \tag{21}$$

the optimal solution of the modified problem $\acute{\mathcal{P}}_B$ can be chosen from the feasible patterns, which are the same as those in (17), to be

$$\acute{\mathbf{K}}_B^o = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{22}$$

Combining $\mathbf{K}_A^o$ and $\acute{\mathbf{K}}_B^o$, a suboptimal feedback pattern is obtained as

$$\mathbf{K}_{AB}^s = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{23}$$

which yields the suboptimal cost c($\mathbf{K}^s$)=c($\mathbf{K}_A^o$) + $c^A(\acute{\mathbf{K}}_B^o)$=5.

On the other hand, if one starts with problem $\mathcal{P}_B$, then among the feasible patterns in (17),

$$\mathbf{K}_B^o = \mathbf{K}^o \tag{24}$$

would be the optimal solution to $\mathcal{P}_B$. With the cost matrix modified accordingly into

$$K_B^c = \begin{bmatrix} 0 & 2 \\ 3 & 5 \end{bmatrix}, \tag{25}$$

the optimal solution of $\acute{\mathcal{P}}_A$ would then be $\acute{\mathbf{K}}^o_A = \mathbf{K}^o$, so that $\mathbf{K}^s_{AB} = \mathbf{K}^o_B + \acute{\mathbf{K}}^o_A = \mathbf{K}^o$.

4.2 Solution of Problem $\mathcal{P}_A(\mathcal{D}, K^c)$

By Theorem 3.5, the condition $\Lambda^A_{\mathbf{K}} = \emptyset$ is equivalent to g.r.($\hat{\mathrm{S}}$)= n + m + r. Therefore, an alternative statement of problem $\mathcal{P}_A$ can be given as

$$\mathcal{P}_A : \left.\begin{array}{ll} min & c(\mathbf{K}) \\ s.t. & gr(\hat{\mathbf{S}}) = n + m + r \end{array}\right\}. \tag{26}$$

In this formulation the constraint is stated in terms of the closed-loop system structure matrix $\hat{\mathbf{S}}$, but the cost involves only a part of $\hat{\mathbf{S}}$, namely $\mathbf{K}$. In order to translate the cost into one involving $\hat{\mathbf{S}}$, we define the system cost matrix $\hat{S}^c$ as

$$\hat{S}^c = (\hat{s}^c_{ij}) = \begin{bmatrix} A^c & B^c & \Gamma^c \\ \Gamma^c & I^c & K^c \\ C^c & \Gamma^c & I^c \end{bmatrix}, \tag{27}$$

where $A^c = (a^c_{ij})$ with

$$a^c_{ij} = \begin{cases} 0 & , \quad \mathbf{a}_{ij} = 1 \\ \gamma & , \quad \mathbf{a}_{ij} = 0 \end{cases}, \tag{28}$$

B^c and C^c are defined similarly ; Γ^c is a matrix of suitable dimension, consisting of all γ's ; and I^c has zero diagonals and γ off-diagonals. In other words, $\hat{S}^c$ is obtained from $\hat{\mathbf{S}}$ by replacing nonzero elements by 0 and zero elements by γ except those of $\mathbf{K}$, which are replaced by the corresponding costs. Now defining the total cost associated with a system structure matrix $\hat{\mathbf{S}}$ to be

$$c(\hat{\mathbf{S}}) = \sum_{\hat{\mathbf{s}}_{ij}=1} \hat{s}^c_{ij} \tag{29}$$

we reformulate problem $\mathcal{P}_A$ as

$$\mathcal{P}_A : \left.\begin{array}{ll} min & c(\hat{\mathbf{S}}) \\ s.t. & gr(\hat{\mathbf{S}}) = n + m + r \end{array}\right\}. \tag{30}$$

Problem $\mathcal{P}_A$ as stated in (30) is known as the Assignment Problem [20], [21], and is equivalent to the following special network flow problem:

Consider a flow network described by a weighted digraph $\mathcal{D} = (\mathcal{V} \bigcup \mathcal{T},\ \mathcal{E},\ w)$, where $\mathcal{V} = \{v_1, v_2, .., v_N\}$ is a set of vertices, each of which is associated with one unit of supply of an item, $\mathcal{T} = \{t_1, t_2, .., t_N\}$ is a set of sinks each of which is associated with one unit of demand of the item, $\mathcal{E}$ is a set of edges of the form (v_j, t_i) which represent the shipping lines from sources to sinks, and w is a real valued function on $\mathcal{E}$ such that $w_{ij} = w(v_j, t_i)$ represents the unit shipping cost along the edge (v_j, t_i). The flow problem is to ship the supply at the sources through the edges to the sinks at minimum cost. Denoting the amount of flow along the edge (v_j, t_i) by f_{ij}, the problem is formulated as

$$\left.\begin{array}{ll} min & \sum_{(v_j,t_i)\in\mathcal{E}} w_{ij} f_{ij} \\ & \\ s.t. & \sum_{j=1}^{N} f_{ij} = \sum_{j=1}^{N} f_{ji} = 1 \ , i = 1, 2, .., N \\ & \\ & f_{ij} \geq 0 \ \ , \ \ (v_j, t_i) \in \mathcal{E} \end{array}\right\} . \tag{31}$$

The flow problem in (31) is a linear one, and can be solved by polynomially bounded algorithms. An efficient solution procedure is the Hungarian Assignment Algorithm, which is repeated below for convenience on the weight matrix $\hat{S}^c$.

Hungarian Assignment Algorithm ([20], [28]):

1. For each row of $\hat{S}^c$, subtract the minimum element of the row from all the elements of the row.

2. Repeat Step 1 for columns of $\hat{S}^c$.

3. Pick maximum number of zeros in $\hat{S}^c$ which lie on different rows and columns. If a zero is picked from every row, go to Step 5.

4. Draw a minimum number of lines (vertical or horizontal) that cover all zeros in $\hat{S}^c$ (the number of lines is the same as the number of zeros picked in Step 3). Find the minimum of all elements which are not covered by these lines; subtract the minimum from the uncovered elements and add to the ones that are covered by both horizontal and vertical lines . Go to Step 3.

5. Optimum solution of problem $\mathcal{P}_A$ is obtained simply by setting

$$\mathrm{k}_{ij}^{oA} = \begin{cases} 1, & if\ a\ zero\ at\ the\ corresponding\ position\ of\ K^c\ is\ picked \\ 0, & otherwise \end{cases} .$$

Before closing the section, we would like to point out that Step 3 of the Hungarian Assignment Problem is the Maximum Transversality Problem [26], and is equivalent to computing the generic rank of a matrix.

4.3 Solution of Problem $\mathcal{P}_B(\mathcal{D}, K^c)$

Considering condition (B) of the Theorem 3.4, we note that if a state vertex x_i occurs in a strong component of the closed-loop system digraph $\hat{\mathcal{D}}$ which contains a feedback edge, then all state vertices that are strongly connected to x_i in the open loop system digraph $\mathcal{D}$ have the same property. Therefore, a condensation of the strong components of $\mathcal{D}$ before inserting the feedback edges does not effect the set of feasible feedback patterns for problem $\mathcal{P}_B$. In other words, $\mathcal{P}_B(\mathcal{D}, K^c)$ is equivalent to $\mathcal{P}_B(\mathcal{D}^*, K^c)$, where $\mathcal{D}^* = (\mathcal{X}^* \cup \mathcal{U} \cup \mathcal{Y}, \mathcal{E}^*)$ is obtained from $\mathcal{D} = (\mathcal{X} \cup \mathcal{U} \cup \mathcal{Y}, \mathcal{E})$ by condensing the strong components. Also, noting that condition (B) of Theorem 3.4 is concerned only with the reachability properties of $\mathcal{D}$, we further modify $\mathcal{D}^*$ to $\mathcal{D}^*_M = (\mathcal{X}^* \cup \mathcal{U} \cup \mathcal{Y}, \mathcal{E}^*)$ which has the same input and output reachability properties. In other words, if $\mathcal{D}^*$ has the reachability matrix

$$\mathbf{R}^* = \begin{bmatrix} \mathbf{F}^* & \mathbf{G}^* & 0 \\ 0 & \mathbf{I} & 0 \\ \mathbf{H}^* & \mathbf{I} & \mathbf{I} \end{bmatrix},$$

$\mathcal{D}^*_M$ is the digraph whose adjacency matrix is

$$\mathbf{S}^*_m = \begin{bmatrix} 0 & \mathbf{G}^* & 0 \\ 0 & 0 & 0 \\ \mathbf{H}^* & 0 & 0 \end{bmatrix}. \tag{32}$$

Note that, the input and output vertices of $\mathcal{D}$ are preserved in $\mathcal{D}^*_M$. Now putting in the feedback edges , we obtain closed-loop modified digraph $\hat{\mathcal{D}}^*_M = (\mathcal{X}^* \cup \mathcal{U} \cup \mathcal{Y}, \mathcal{E}^*_M \cup \mathcal{E}_K)$, which is much simpler than $\hat{\mathcal{D}}$, but is equivalent to $\hat{\mathcal{D}}$ as far as problem $\mathcal{P}_B$ is concerned. The two steps involved in obtaining $\mathcal{D}^*_M$ from $\mathcal{D}$ are illustrated in Figs. 4.3 and 4.4, for a simple digraph shown in Fig. 4.2.

Considering conditon (B) of Theorem 3.4 applied to $\hat{\mathcal{D}}^*_M$, we observe that a feasible feedback pattern is one which allows every state vertex of $\hat{\mathcal{D}}^*_M$ to reach itself through a path which includes at least one feedback edge. This observation allows for a reformulation of problem $\mathcal{P}_B$ also as a network flow problem:

Consider a flow network described by a digraph $(\mathcal{X}^*_s \cup \mathcal{Y} \cup \mathcal{U} \cup \mathcal{X}^*_t,\ \mathcal{E}_{yx} \cup \mathcal{E}_{uy} \cup \mathcal{E}_{xu})$, where $\mathcal{X}^*_s = \{x^*_{s1}, x^*_{s2}, .., x^*_{sN}\}$ is a set of sources (sending points) ; $\mathcal{X}^*_t = \{x^*_{t1}, x^*_{t2}, .., x^*_{tN}\}$ is a set of sinks ; $\mathcal{Y}$ and $\mathcal{U}$, which are the same as in $\mathcal{D}$ or

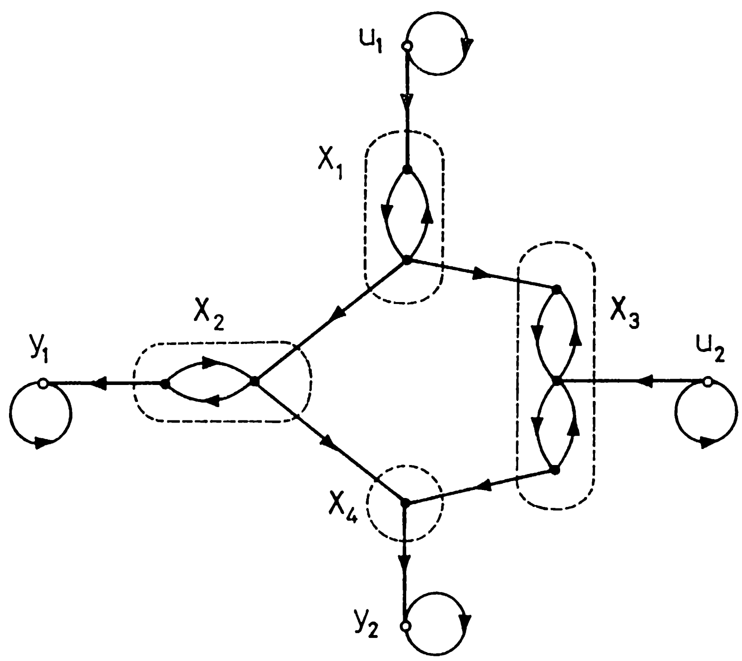


Figure 4.2: A Digraph for condensation example

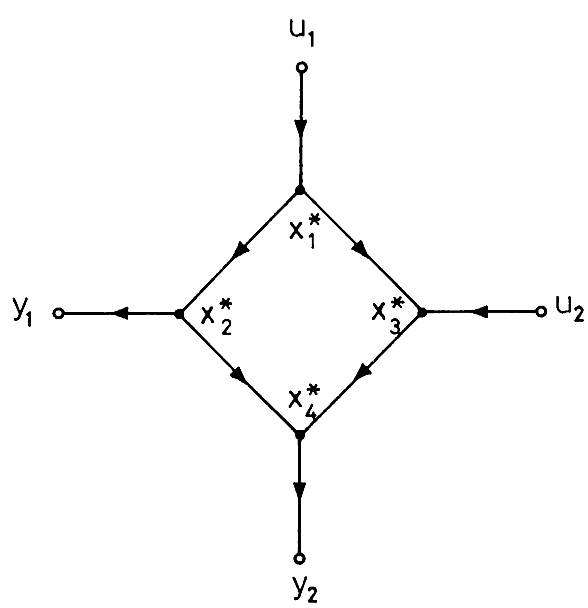


Figure 4.3: Condensed digraph of the network in Fig. 4.2

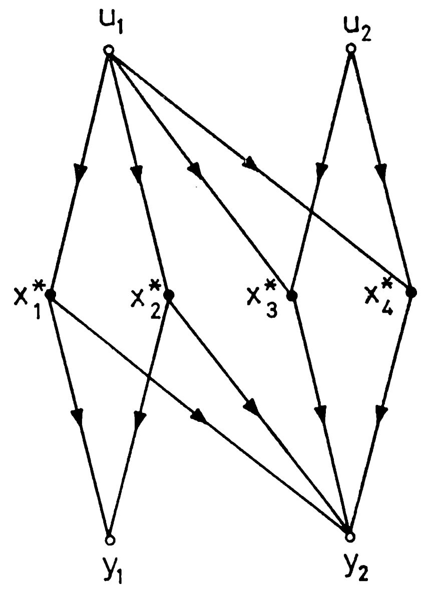


Figure 4.4: Modified digraph of the network in Fig. 4.2

$\mathcal{D}^*_M$, are the sets of intermediate vertices ; $\mathcal{E}_{yx}$ is a set of edges of the form (x^*_{sj}, y_i) corresponding to the edges (x^*_j, y_i) of $\mathcal{D}^*_M$; $\mathcal{E}_{uy}$=$\mathcal{E}_K$; and finally $\mathcal{E}_{xu}$ contains edges of the form (u_j, x^*_{ti}) corresponding to $(u_j, x^*_i) \in \mathcal{E}^*_M$. The flow network corresponding to $\hat{D}^*_M$ of Fig. 4.4 is shown Fig. 4.5.

As usual, each x^*_{si} contains one unit of supply of an item to be shipped to an x^*_{tj}, each of which demands one unit of the item. Shipping cost through the edges in $\mathcal{E}_{yx}$ and $\mathcal{E}_{xu}$ is zero. However, unlike the linear flow problem of the previous section, shipping cost through a feedback edge (y_j, u_i) is k^c_{ij}, which is fixed irrespective of the flow unless it is zero. This is due to the fact that the same feedback edge may be used to satisfy condition (B) of Theorem 3.4 for more than one state vertices at the same cost. Denoting the flows through the edges (x^*_{sj}, y_i), (y_j, u_i) and (u_j, x^*_{ti}) by g_{ij}, f_{ij} and h_{ij} respectively, the flow problem can be formulated as

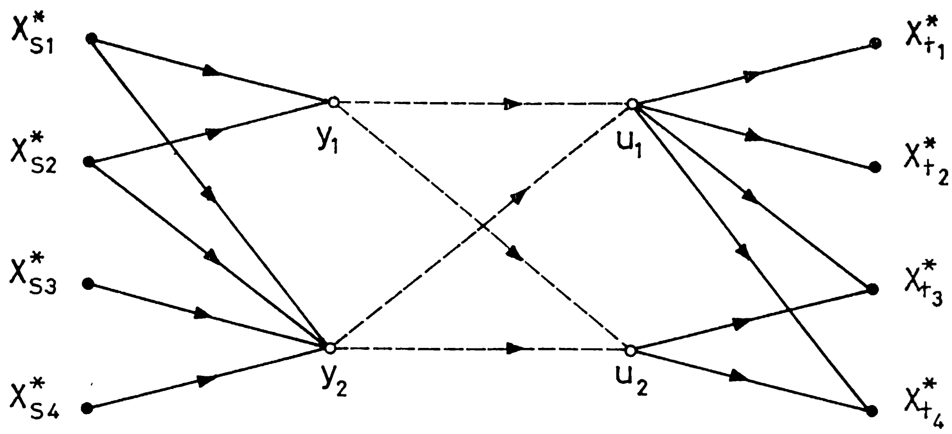


Figure 4.5: The flow network corresponding to the modified digraph in Fig. 4.4

$$
\mathcal{P}_B: \quad \left.\begin{array}{ll}
min & \sum_{i=1}^{m} \sum_{j=1}^{r} w(f_{ij}) \\
s.t. & \sum_{j=1}^{r} g_{ji} = \sum_{j=1}^{m} h_{ij} = 1 \ , \quad i = 1, 2, .., N \\
 & \sum_{j=1}^{N} g_{ij} = \sum_{j=1}^{m} f_{ji} \ , \quad i = 1, 2, .., r \\
 & \sum_{j=1}^{r} f_{ij} = \sum_{j=1}^{N} h_{ji} \ , \quad i = 1, 2, .., m \\
 & f_{ij}, g_{ij}, h_{ij} \geq 0
\end{array}\right\} \tag{33}
$$

where

$$
w(f_{ij}) = \begin{cases} k^c_{ij} \ , & f_{ij} > 0 \\ 0 \ , & f_{ij} = 0 \end{cases} . \tag{34}
$$

Once an optimal solution $\{f^o_{ij}\}$ to the above flow problem is found, the optimal solution of the original problem can readily be obtained as $\mathbf{K}^o_B = (\mathbf{k}^{oB}_{ij})$, where $\mathbf{k}^{oB}_{ij} = \mathbf{1}$ if and only if $f^o_{ij} > 0$.

Due to the nature of the cost function w in (34), the flow problem in (33) is a nonconvex, nonlinear problem, which is usually solved by branch-and-bound algorithms [29]. We can, however, convert this flow problem to a *generalized assignment problem*, which enables us to develop an implicit enumeration algorithm for the solution. For this purpose we define a block cost matrix K_G^c as

$$K_G^c = \tilde{W} = [\tilde{W}_{pq}]_{N\times N}, \tag{35}$$

where each block $\tilde{W}_{pq} = (\tilde{w}_{ij}^{pq})_{m\times r}$ of W is associated with a pair of state vertices (x_q^*, x_p^*), and has the elements

$$\tilde{w}_{ij}^{pq} = \begin{cases} k_{ij}^c, & if\ u_i\ reaches\ x_p^*\ and\ x_q^*\ reaches\ y_j\ in\ \mathcal{D}^* \\ \gamma, & otherwise \end{cases} \tag{36}$$

In other words, $\tilde{w}_{ij}^{pq}$ is the cost of a path from x_q^* to x_p^* through y_j and u_i, which is infinity if no such path exists. It is clear that condition (B) is equivalent to picking N elements in $\tilde{W}$, which are located in different block rows and columns of $\tilde{W}$. Now, $\mathcal{P}_B$ can be reformulated as

$$\mathcal{P}_B : \left.\begin{array}{ll} min & c(\mathbf{K}) \\ s.t. & block\ gr(\mathbf{K}_G) = N, \end{array}\right\} \tag{37}$$

where

$$block\ gr(\mathbf{K}_G) = block\ gr(\tilde{\mathbf{W}}) \stackrel{\text{def}}{=} gr(\mathbf{W}) \tag{38}$$

with $\mathbf{W} = (\mathbf{w}_{pq})_{N\times N}$ defined as

$$\mathbf{w}_{pq} = \begin{cases} 1, & if\ \tilde{w}_{ij}^{pq} \neq \gamma\ for\ some\ (i,j) \\ 0, & otherwise. \end{cases} \tag{39}$$

Fig. 4.6 shows two feasible generalized assignments on the generalized cost matrix of the network of Fig. 4.5.

We now present an implicit enumeration algorithm for the solution of $\mathcal{P}_B$ in (37). For this purpose we first introduce the following notation:

Let the feedback edges and the corresponding costs be renamed as e_1 , e_2, ..., e_s and c_1, c_2, ..., c_s, where s=mr such that to any e_l, $1 \leq l \leq s$ there corresponds a unique feedback edge (y_j, u_i), $1 \leq i \leq m$, $1 \leq j \leq r$, and $c_l = k_{ij}^c$. Associated with the set $\{e_l\}$, we have an s-dimensional array J whose elements take four distinct values, 1,0, F and U. $J(l)$=1 or 0 means the corresponding feedback edge e_l is or is not included in the current pattern, $J(l) = F$ indicates that e_l may later be included into the pattern, i.e., e_l is free, and finally $J(l) = U$ means addition of e_l results in a feedback pattern whose cost is no smaller than the current optimum, i.e.,e_l is useless at that step. The cost of a

		x^*_{S1} y_1	x^*_{S1} y_2	x^*_{S2} y_1	x^*_{S2} y_2	x^*_{S3} y_1	x^*_{S3} y_2	x^*_{S4} y_1	x^*_{S4} y_2
x^*_{t1}	u_1	k_{11}	k_{12}	k_{11}	k_{12}	γ	k_{12}	γ	k_{12}
	u_2	γ	γ	γ	γ	γ	γ	γ	γ
x^*_{t2}	u_1	k_{11}	k_{12}	k_{11}	k_{12}	γ	k_{12}	γ	k_{12}
	u_2	γ	γ	γ	γ	γ	γ	γ	γ
x^*_{t3}	u_1	k_{11}	k_{12}	k_{11}	k_{12}	γ	k_{12}	γ	k_{12}
	u_2	k_{21}	k_{22}	k_{21}	k_{22}	γ	k_{22}	γ	k_{22}
x^*_{t4}	u_1	k_{11}	k_{12}	k_{11}	k_{12}	γ	k_{12}	γ	k_{12}
	u_2	k_{21}	k_{22}	k_{21}	k_{22}	γ	k_{22}	γ	k_{22}

(a)

		x^*_{S1} y_1	x^*_{S1} y_2	x^*_{S2} y_1	x^*_{S2} y_2	x^*_{S3} y_1	x^*_{S3} y_2	x^*_{S4} y_1	x^*_{S4} y_2
x^*_{t1}	u_1	k_{11}	k_{12}	k_{11}	k_{12}	γ	k_{12}	γ	k_{12}
	u_2	γ	γ	γ	γ	γ	γ	γ	γ
x^*_{t2}	u_1	k_{11}	k_{12}	k_{11}	k_{12}	γ	k_{12}	γ	k_{12}
	u_2	γ	γ	γ	γ	γ	γ	γ	γ
x^*_{t3}	u_1	k_{11}	k_{12}	k_{11}	k_{12}	γ	k_{12}	γ	k_{12}
	u_2	k_{21}	k_{22}	k_{21}	k_{22}	γ	k_{22}	γ	k_{22}
x^*_{t4}	u_1	k_{11}	k_{12}	k_{11}	k_{12}	γ	k_{12}	γ	k_{12}
	u_2	k_{21}	k_{22}	k_{21}	k_{22}	γ	k_{22}	γ	k_{22}

(b)

Figure 4.6: Two feasible generalized assignments for the network of Fig. 4.5

feedback pattern described by the current form of J is denoted by $c(J)$, and is computed as

$$c(J) = \sum_{J(l)=1} c_l. \tag{40}$$

Finally , the current minimum is denoted by c^*, and the current best pattern by J^*.

Implicit Enumeration Algorithm:

0. (Initialization) Set $J(l) = F$, $1 \leq l \leq s$, $c^* = \gamma$, $\mathbf{k}^o_{ij} = \mathbf{1}$, $1 \leq i \leq m$, $1 \leq j \leq l$.

1. Find L=min$\{l \ : J(l) = F\}$. Set $J(L)$=1.

2. Check J for feasibility (subroutine). If J is feasible, set J^*=J, $c^* = c(J)$, and go to Step 5. Otherwise proceed to the next step.

3. For $1 \leq l \leq s$ and $J(l) = F$, if $c(J) + c_l \geq c^*$, set $J(l) = U$. If there remains any l with $J(l) = F$ proceed to the next step. Otherwise go to Step 5.

4. Check J for potential feasibility (subroutine) . If J is potentially feasible, go to Step 1. Otherwise proceed to the next step.

5. If $J(l) \neq 1$ for all $1 \leq l \leq s$, proceed to the next step. Otherwise, find $L = max\{\ L\ :\ J(l) = 1\ \}$, set $J(L) = 0$, $J(l) = F$ for $l > L$, and go to Step 3.

6. For $1 \leq l \leq s$ and $J^*(l) \neq 1$, find the unique pair of indices (i, j) defined by l , and set $\mathbf{k}_{ij}^{oB}$=0. $\mathbf{K}^{oB} = (\mathbf{k}_{ij}^{oB})$ is an optimum solution of $\mathcal{P}_B$.

Subroutine For Feasibility and Potential Feasibility

1. Construct $\mathbf{W} = (\mathbf{w}_{pq})_{N \times N}$ corresponding to J as follows:

i). Initially $\mathbf{w}_{pq} = 0$, $1 \leq p, q \leq N$.

ii). For $1 \leq l \leq s$ and $J(l) = 1$ for feasibility , or $J(l) = 1$ or F for potential feasibility, find the unique pair of indices (i, j) defined by l. For all $1 \leq p, q \leq N$ and $\tilde{w}_{ij}^{pq} \neq \gamma$, set $\mathbf{w}_{pq} = 1$.

2. If gr(**W**)=N , J is feasible (or potentially feasible).

It is observed that the Implicit Enumeration Algorithm is also a branch-and-bound algorithm, which scans a binary tree starting from the root $(FF...F)$. The two immediate successors of any node $(**...*FF...F)$, where $*$ is either 1 or 0, are $(**...*1F...F)$ and $(**...*0F...F)$. The algorithm is a depth-first search [30] on the tree, where a branch is bounded when either the pattern corresponding to its root is a feasible one with a smaller cost, or all the subsequent patterns are infeasible or have higher costs.

We note that the proposed implicit enumeration algorithm can be improved considerably if more attention is paid to the choice of the feedback edge to be included into the current pattern in Step 1. Rather than picking the first edge marked F in the sequence, the decision may be based on other criteria. Below we suggest few alternatives in the order of increasing complexity:

a) Choose L such that $c_L = min\{\ c_l :\ \ J(l) = F\}$.

b) For $1 \leq l \leq s$ with $J(l)$ =1, mark the blocks of $\tilde{W}$ in which the feedback edge e_l appears. For $1 \leq l \leq s$ with $J(l) = F$, let n_l be the number of unmarked blocks of $\tilde{W}$, in which the edge e_l appears. Choose L such that $n_L = max\{\ n_l :\ \ J(l) = F\}$.

c) Choose L such that $c_L/n_L = min\{\ c_l/n_l\ \ \ : J(l) = F\}$, where n_l is defined as in (b) above.

In case (a), simply the cheapest free feedback edge is added to the current pattern. In case (b) , that edge having the potential of providing maximum increase in the generic rank of the test matrix **W** is preferred. Case (c) is a combination of both criteria, which favors the edge that costs least for a potential unit increase in the generic rank. Note, however, that in case the edges are included into the current pattern in an order other than the natural order, Step 5 of the algorithm should be modified so as to complement the last edge included in the pattern and to free the subsequent edges in the order.

4.4 A Special Case

Although the implicit enumeration algorithm of the previous section provides an optimum solution to the problem $\mathcal{P}_B(\mathcal{D}^*, K^c)$, in the worst case it may have to go through all possible feedback patterns before the solution is reached. In this section, we speculate on an idea of translating the nonlinear problem $\mathcal{P}_B$ to a linear problem $\mathcal{P}_A$ by constructing a modified digraph $\mathcal{D}_M$ and a modified cost matrix K_M^c such that optimum solutions of $\mathcal{P}_B(\mathcal{D}^*, K^c)$ and $\mathcal{P}_A(\mathcal{D}_M, K_M^c)$ are the same. The idea is motivated by the fact that if a set of feasible solutions to $\mathcal{P}_B(\mathcal{D}^*, K^c)$ which contains the optimum solution $\mathbf{K}_B^o$ were known, then one could always and easily construct $\mathcal{D}_M$ and K_M^c such that $\mathcal{P}_A(\mathcal{D}_M, K_M^c)$ admits $\mathbf{K}_B^o$ as optimum solution. In the following, we show that for a class of digraphs $\mathcal{D}^*$, $\mathcal{P}_A(\mathcal{D}_M, K_M^c)$ can be constructed without knowing the optimum solution.

We start by defining the index sets I_l and J_l associated with the state vertices of $\mathcal{D}^*$ as

$I_l = \{i : u_i \textit{ reaches } x_k^* \textit{ in } \mathcal{D}^*\}$, $J_l = \{j : x_k^* \textit{ reaches } y_j \textit{ in } \mathcal{D}^*\}$, $l = 1, 2, .., N$.

Suppose that there exists a permutation $(l_1, l_2, ..., l_N)$ of the state vertices X^* of $\mathcal{D}^*$ such that

$$I_{l_p} \subseteq I_{l_{p+1}}, \; J_{l_p} \supseteq J_{l_{p+1}}, \; p = 1, 2, ..., N-1. \tag{41}$$

Considering the reachability matrix

$$\mathbf{R}^* = \begin{bmatrix} \mathbf{F}^* & \mathbf{G}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m & \mathbf{0} \\ \mathbf{H}^* & \mathbf{0} & \mathbf{I}_r \end{bmatrix}$$

of $\mathcal{D}^*$, it is easy to see that condition (41) is equivalent to existence of permutation matrices $\mathbf{P}_x$, $\mathbf{P}_u$, and $\mathbf{P}_y$ such that $\mathbf{P}_x^T\mathbf{G}^*\mathbf{P}_u$ and $\mathbf{P}_y\mathbf{H}^*\mathbf{P}_x$ have the following structures.

$$\mathbf{P}_x^T\mathbf{G}^*\mathbf{P}_u = \begin{array}{c} \left[\begin{array}{cc} & \mathbf{0} \\ \ddots & \\ \mathbf{1} & \end{array}\right] \end{array} \begin{array}{l} \rightarrow s_1 \\ \\ \rightarrow s_M \\ \\ \end{array} \tag{42}$$

$$\mathbf{P}_y\mathbf{H}^*\mathbf{P}_x = \begin{array}{c} \begin{array}{cc} t_1 & t_R \\ \downarrow & \downarrow \end{array} \\ \left[\begin{array}{cc} & \mathbf{0} \\ \ddots & \\ \mathbf{1} & \end{array}\right] \end{array}, \tag{43}$$

where $\mathbf{1}$ indicates that the region is filled with 1's. Let the sets of input and output vertices of $\mathcal{D}^*$ be partitioned in accordance with the structures of $\mathbf{P}_x^T\mathbf{G}^*\mathbf{P}_u$ and $\mathbf{P}_y\mathbf{H}^*\mathbf{P}_x$ as

$$\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \ldots \cup \mathcal{U}_M, \quad \mathcal{Y} = \mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \ldots \cup \mathcal{Y}_R. \tag{44}$$

We now construct an intermediate digraph $\mathcal{D}_I^* = (\mathcal{X}_I^* \cup \mathcal{U}_I^* \cup \mathcal{Y}_I^*, \mathcal{E}_I^*)$, which is characterized by the adjacency matrix

$$\mathbf{S}_I^* = \begin{bmatrix} \mathbf{A}_I^* & \mathbf{B}_I^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{C}_I^* & \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{45}$$

where

$$\mathbf{A}_I^* = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}_{N\times N},$$

$$\mathbf{B}_I^* = \left[\begin{array}{ccc} \begin{array}{c}1\\0\\ \vdots\\0\end{array} & & 0 \\ & \ddots & \\ 0 & & \begin{array}{c}1\\0\\ \vdots\\0\end{array}\end{array}\right] \begin{array}{l} \to s_1 \\ \\ \\ \\ \\ \to s_M \\ \\ \\ \\ \end{array}$$

$$\mathbf{C}_I^* = \begin{array}{c} \begin{array}{ccc} \quad\quad\quad t_1 & & \quad\quad\quad t_R \\ \quad\quad\quad \downarrow & & \quad\quad\quad \downarrow \end{array} \\ \left[\begin{array}{ccc} 0 \ldots 0 \; 1 & & 0 \\ & \ddots & \\ 0 & & 0 \ldots 0 \; 1 \end{array}\right] \end{array}.$$

Note that, the state vertices of $\mathcal{D}_I^*$ are arranged in the form of a chain, each group of inputs of $\mathcal{D}^*$ belonging to the same input set $\mathcal{U}_p$ is replaced by a single input u_p^*, and similarly each group of output vertices belonging to the same output set $\mathcal{Y}_q$ is replaced by a single output y_q^* in $\mathcal{D}_I^*$.

We further define a modified cost matrix as $K_M^c = (k_{pq}^{cM})_{M\times R}$, where

$$k_{pq}^{cM} = \begin{cases} min\{k_{ij}^c \ : \ u_i \in \mathcal{U}_p, \ y_j \in \mathcal{Y}_q\} & , \quad if \ s_p \leq t_q \\ \gamma & , \quad otherwise \end{cases},$$

and state the following:

Theorem 4.1: To every optimal solution of $\mathcal{P}_B(\mathcal{D}^*, K^c)$ there corresponds an optimal solution of $\mathcal{P}_B(\mathcal{D}_I^*, K_M^c)$ with the same cost and vice versa.

Proof: Let an optimum solution $\mathbf{K}_B^o$ of $\mathcal{P}_B(\mathcal{D}^*, K^c)$ correspond to a subset of feedback edges $\mathcal{E}_K^{o*} \subset \mathcal{E}_K^*$. For a feedback edge $(y_j, u_i) \in \mathcal{E}_K^{o*}$, if $u_i \in \mathcal{U}_p$ and $y_j \in \mathcal{Y}_q$, then $s_p \leq t_q$ and $k_{ij}^c = min\{k_{i'j'}^c : u_{i'} \in \mathcal{U}_p, \ y_{j'} \in \mathcal{Y}_q\}$. Now construct a set of feedback edges $\mathcal{E}_{IK}^{o*}$ in $\mathcal{D}_I^*$ as follows:

$\mathcal{E}_{IK}^{o*} = \{(y_q^*, u_p^*) \in \mathcal{E}_{IK}^* : (y_j, u_i) \in \mathcal{E}_K^{o*} \ \ for \ \ some \ \ u_i \in \mathcal{U}_p, \ y_j \in \mathcal{Y}_q\}$. Then obviously, $\mathcal{E}_{IK}^{o*}$ corresponds to an optimum feedback pattern $\mathbf{K}_{IB}^o$ for $\mathcal{P}_B(\mathcal{D}_I^*, K_M^c)$ with $c(\mathbf{K}_{IB}^o) = c(\mathbf{K}_B^o)$.

Conversely, if $\mathcal{E}_{IK}^{o*}$ corresponds to an optimum solution of $\mathcal{P}_B(\mathcal{D}_I^*, K_M^c)$, then $\mathcal{E}_K^{o*}$ defined as

$$\mathcal{E}_K^{o*} = \{(y_j, u_i) \in \mathcal{E}_K^* : u_i \in \mathcal{U}_p,\ y_j \in \mathcal{Y}_q \ \ for \ \ some \ \ (y_q^*, u_p^*) \in \mathcal{E}_{IK}^{o*}\}$$

and

$$k_{ij} = min\{k_{i'j'}^c : u_{i'} \in \mathcal{U}_p,\ y_{j'} \in \mathcal{Y}_q\},$$

corresponds to an optimum solution $\mathcal{P}_B(\mathcal{D}^*, K^c)$ having the same cost.

Because of Theorem 4.1, problem $\mathcal{P}_B(\mathcal{D}^*, K^c)$ can safely be replaced by $\mathcal{P}_B(\mathcal{D}_I^*, K_M^c)$. At this point it should be noted that $\mathcal{P}_B(\mathcal{D}_I^*, K_M^c)$ is already much simpler than $\mathcal{P}_B(\mathcal{D}^*, K^c)$ because of the special structure of $\mathcal{D}_I^*$ and the fact that K_M^c contains fewer non-γ entries than K^c does (which speeds up steps 3 and 4 of the implicit enumeration algorithm).

The next step is to construct $\mathcal{D}_M$ from $\mathcal{D}_I^*$ such that optimum solutions of $\mathcal{P}_B(\mathcal{D}_I^*, K_M^c)$ and $\mathcal{P}_A(\mathcal{D}_M, K_M^c)$ coincide. For this purpose, we first state the following:

Theorem 4.2:

a) Each feedback edge (y_q^*, u_p^*), $s_p \leq t_q$, defines a unique cycle in the closed loop digraph $\hat{\mathcal{D}}_I^*$, which covers the state vertices x_i^*, $s_p \leq i \leq t_q$,

b) A given feedback pattern is feasible for $\mathcal{P}_B(\mathcal{D}_I^*, K_M^c)$ if and only if the family of cycles defined by the corresponding feedback edges cover all the state vertices.

c) A state vertex x_i^* can be common to at most two cycles in a cycle family of $\mathcal{D}_I^*$ defined by an optimal feedback pattern.

d) A state vertex x_i^* can occur in two different cycles only if there exist at least two inputs that reach x_i^*, and at least two outputs that x_i^* reach, i.e. $s_2 \leq i \leq t_{R-1}$.

Proof: The statements (a), (b) and (d) are obvious. To prove (c), assume that x_i^* is common to k cycles in a cycle family defined by a feasible pattern. Let the feedback edges that define these cycles be $(y_{q_1}^*, u_{p_1}^*), ..., (y_{q_k}^*, u_{p_k}^*)$, with $p_1 < p_2 < ... < p_k \leq i$. If the feasible pattern is an optimal one, then we should also have $i \leq q_1 < q_2 < ... < q_k$, because if $q_l < q_{l-1}$ for some $l = 2, 3, ..., k$, then the feedback edge $(y_{q_l}^*, u_{q_l}^*)$ can be removed from cycle family to obtain another feasible pattern with a smaller cost. Now it is easy to see that all the state vertices covered by this cycle family are also covered by the two cycles defined by the feedback edges $(y_{q_1}^*, u_{p_1}^*)$ and $(y_{q_k}^*, u_{p_k}^*)$, so that in an optimal pattern each x_i^* is covered by at most two cycles.

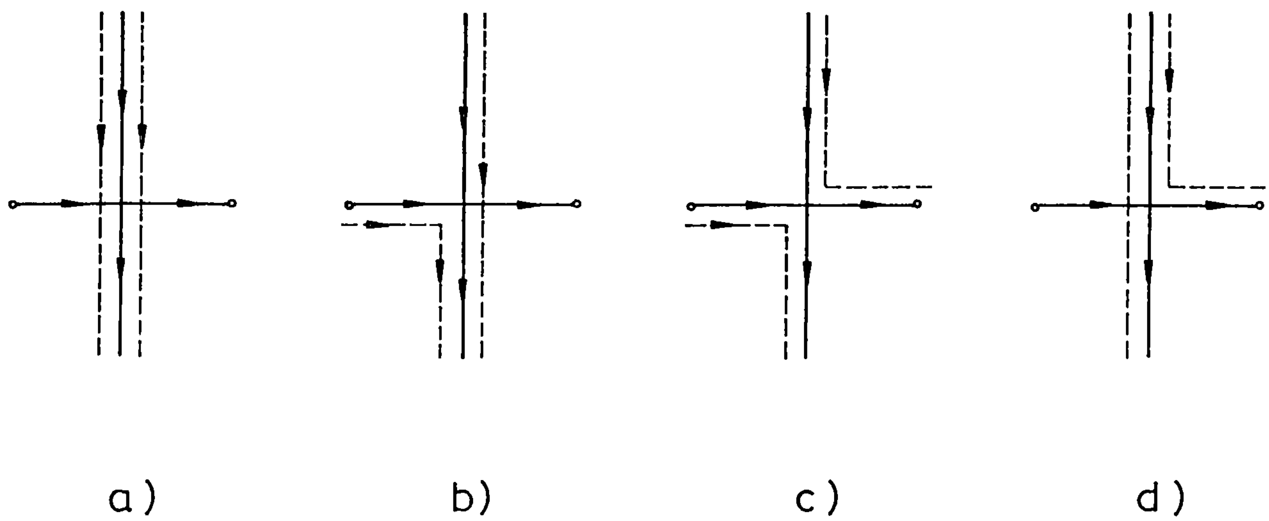


Figure 4.7: Joint cycles covering a vertex

As a result of Theorem 4.2, we conclude that $\mathcal{P}_B(\mathcal{D}_I^*, K_M^c)$ is also a state vertex covering problem. However, unlike problem $\mathcal{P}_A$, the cycles that cover the state vertices of $\mathcal{D}_I^*$ need not be disjoint. This difficulty can be overcome, by expanding the state vertices of $\mathcal{D}_I^*$. Before considering the expansion process, we first note that if a state vertex x_i^* is covered by two different cycles in an optimal pattern, then one of the four situations shown in Fig. 4.7 can occur. In the figure, dashed lines indicate the parts of the covering cycle. It is assumed that x_i^* has an input vertex and an output vertex connected to it. If this is not the case, then either (b) and (c) or (c) and (d) alternatives are eliminated.

By the expansion procedure we aim at expanding each state vertex to a block structure so that the joint cycles in Fig. 4.7 correspond to disjoint cycles in each block in the expanded network. For this purpose, to each x_i^* for which $s_2 \leq i \leq t_{R-1}$, we associate a pair of integers $k_{in,i}$ and $k_{out,i}$ such that

$$k_{in,i} = \begin{cases} 3, & if\ i = s_p\ for\ some\ 3 \leq p \leq M \\ 2, & otherwise \end{cases}$$

$$k_{out,i} = \begin{cases} 3, & if\ i = t_q\ for\ some\ 1 \leq q \leq R-2 \\ 2, & otherwise \end{cases}$$

To explain what $k_{in,i}$ and $k_{out,i}$ means, consider a state vertex x_i^* in $\mathcal{D}_I^*$ with $i = s_p$ for some $3 \leq p \leq M$, that is, x_i^* is reachable by p inputs $u_1^*, ..., u_p^*$

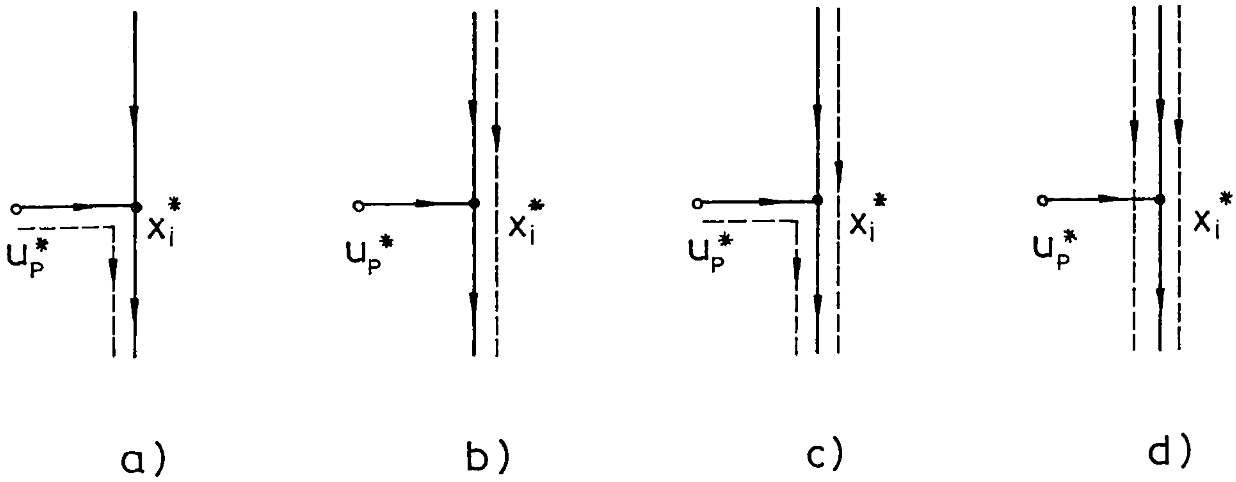


Figure 4.8: A vertex, which is covered by joint cycles from different directions

with u_p^* being adjacent to x_i^*. In an optimal feedback pattern, x_i^* is either covered by a single cycle as demonstrated in Figs. 4.8 (a) and (b), or by two cycles as demonstrated in Figs. 4.8 (c) and (d). In other words, the covering cycle(s) can enter x_i^* in one or two of the three possible directions, which is indicated by $k_{in,i} = 3$. If $i = s_2$ or $s_p < i < s_{p+1}$ for some $p = 2, 3, ..., M-1$, then the cycles in an optimal pattern can enter x_i^* in at most two directions. A similar interpretation can be given for $k_{out,i}$.

We now replace each state vertex x_i^*, $s_2 \le i \le t_{R-1}$ by an expansion block which has an adjacency matrix of the form

$$\mathbf{A}_{ii}^M = \begin{array}{cc} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} & \begin{array}{l} k_{in,i} + k_{out,i} \\ k_{in,i} + k_{out,i} - 1 \end{array} \end{array}$$

The digraph having $\mathbf{A}_{ii}^M$ as its adjacency matrix is shown in Fig. 4.9 for the case $k_{in,i} = 3$ and $k_{out,i} = 2$.

The matrix $\mathbf{A}_{ii}^M$ and the corresponding digraph has the following properties:

a) The generic rank defect of $\mathbf{A}_{ii}^M$ is exactly one,

b) If one or more of the $k_{out,i}$ vertices with outgoing edges are connected by external edges to one or more of the $k_{in,i}$ vertices with incoming edges, then the resulting digraph is covered by a disjoint family of cycles. Two typical cases are illustrated in Fig. 4.10.

With the vertices x_i^* of $\mathcal{D}_I^*$ expanded as explained above, we can now

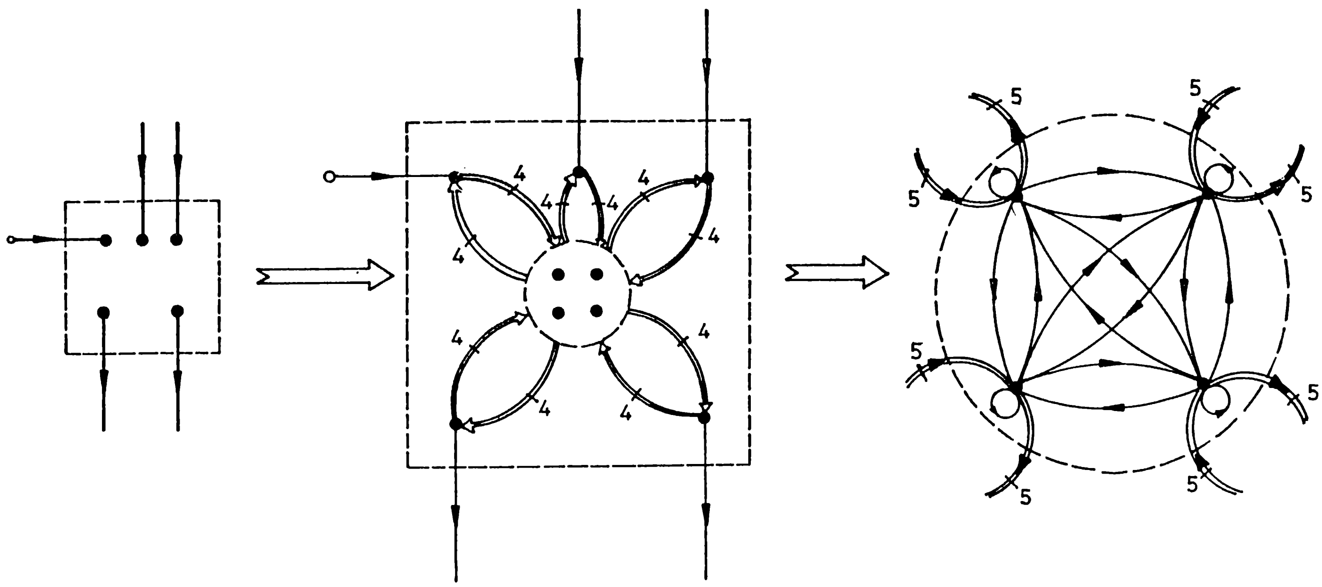
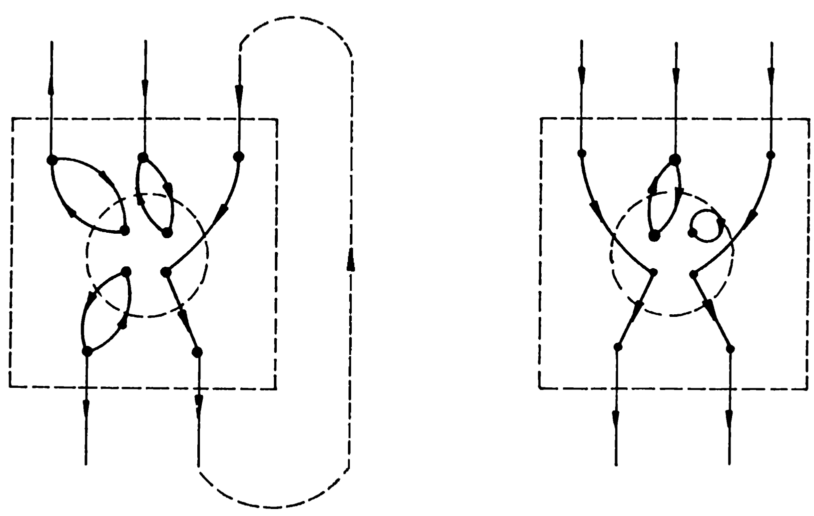


Figure 4.9: Expansion of a vertex



a) b)

Figure 4.10: Covering of the digraph of $\mathbf{A}_{ii}^{M}$ by disjoint cycles

describe the modified digraph $\mathcal{D}_M = (\mathcal{X}_M \cup \mathcal{U}^* \cup \mathcal{Y}^*, \mathcal{E}_M)$ by the adjacency matrix

$$\mathbf{S}_M = \begin{bmatrix} \mathbf{A}_M & \mathbf{B}_M & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_M & \mathbf{0} \\ \mathbf{C}_M & \mathbf{0} & \mathbf{I}_R \end{bmatrix},$$

where $\mathbf{A}_M = (\mathbf{A}^M_{ij})_{N \times N}$, whose blocks are defined as follows:

$$\begin{aligned} \mathbf{A}^M_{ii} &= 0 && , i < s_2 \;\; or \;\; i > t_{R-1} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{array}{l} k_{in,i} + k_{out,i} \\ k_{in,i} + k_{out,i} - 1 \end{array} && , s_2 \leq i \leq t_{R-1}, \end{aligned}$$

$$\mathbf{A}^M_{i+1,i} = 1 \qquad\qquad i < s_2 \;\; or \;\; i > t_{R-1}$$

$$= \begin{array}{c} \downarrow \qquad\qquad \downarrow \\ \begin{array}{c} \rightarrow \\ \\ \\ \rightarrow \\ \\ \end{array} \left[\begin{array}{c|cc|c|c} 0 & 0 & 0 & 0 & \\ \hline 0 & 0 & \mathbf{I}_2 & 0 & \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \\ \hline \multicolumn{4}{c|}{0} & 0 \end{array}\right] \begin{array}{l} \\ \Big\} \; k_{in,i+1} \\ \\ \Big\} \; k_{out,i+1} \\ \Big\} \; k_{in,i+1} + k_{out,i+1} - 1 \end{array} \\ \underbrace{\qquad}_{k_{in,i}} \; \underbrace{\qquad\quad}_{k_{out,i}} \; \underbrace{\qquad\quad}_{k_{in,i} + k_{out,i} - 1} \end{array} , \qquad s_2 \leq i \leq t_{R-1}$$

where the indicated rows and columns need not exist if x^*_i or x^*_{i+1} has no inputs and/or outputs connected to them. Finally,

$$\mathbf{A}^M_{ji} = 0, \; j \neq i, \; i+1,$$

and $\mathbf{B}^M$ and $\mathbf{C}^M$ are similar to $\mathbf{B}^I$ and $\mathbf{C}^I$ with dimensions modified as necessary.

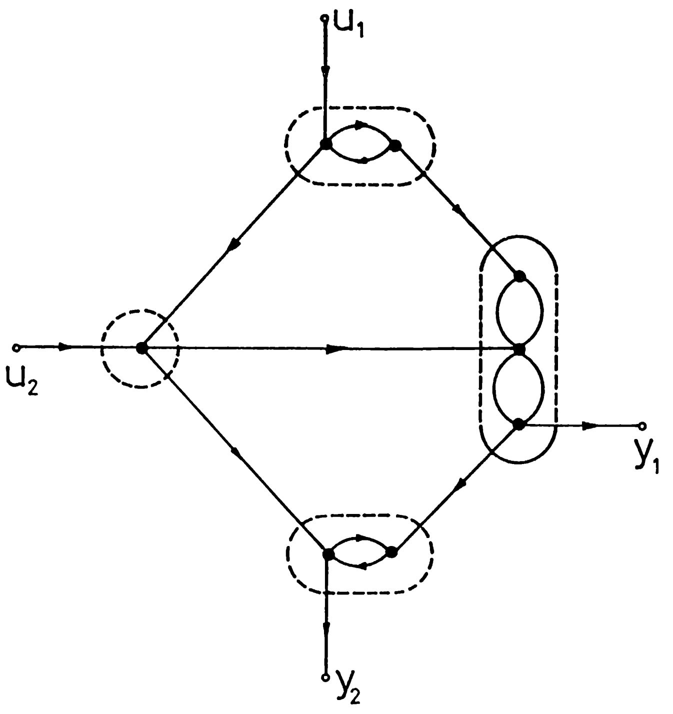


Figure 4.11: Digraph of the system in Example 4.2

We now state the following; whose proof follows directly from the construction of $\mathcal{D}_M$.

Theorem 4.3: To every optimal solution of $\mathcal{P}_B(\mathcal{D}_I^*, K^c)$ there corresponds an optimal solution of $\mathcal{P}_A(\mathcal{D}_M, K_M^c)$, and vice versa.

The steps involved in the construction of $\mathcal{D}_M$ is explained by an example below.

Example 4.2

Consider the digraph $\mathcal{D}$ shown in Fig. 4.11, where the cost matrix is given as

$$K^c = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}.$$

Feasible feedback patterns for $\mathcal{D}$ can be obtained by inspection to be

$$\begin{bmatrix} \mathbf{1} & \star \\ \star & \mathbf{1} \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} \star & \mathbf{1} \\ \star & \star \end{bmatrix} \tag{46}$$

where $\star$ denotes either a 0 or 1. Thus the optimum feedback pattern is

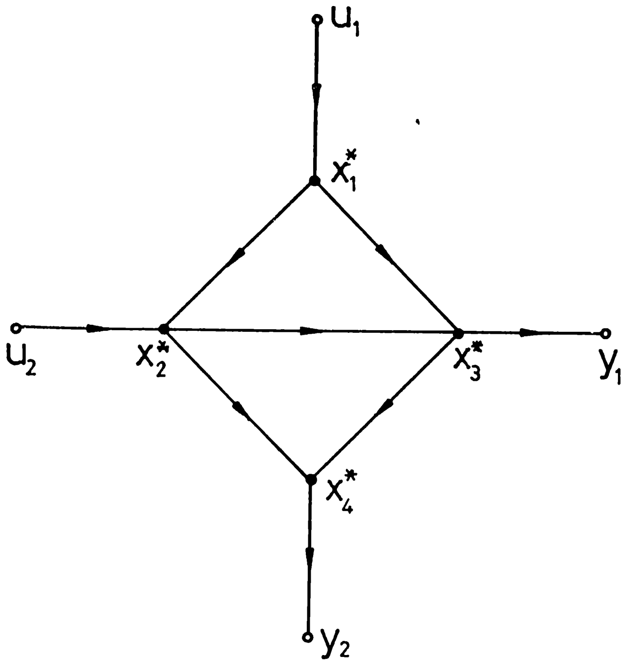


Figure 4.12: Condensation of the digraph in Fig. 4.11

$$\mathbf{K}^o = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

resulting in $c(\mathbf{K}^o) = 4$.

Now consider a decomposition of $\mathcal{P}(\mathcal{D}, K^c)$ into $\mathcal{P}_A(\mathcal{D}, K^c)$ and $\mathcal{P}_B(\mathcal{D}, K^c)$. The optimum solution of $\mathcal{P}_A(\mathcal{D}, K^c)$ is obtained directly by applying the Hungarian Assignment Algorithm to $\hat{\mathcal{S}}^c$ as

$$\mathbf{K}_A^o = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

To illustrate the solution of $\mathcal{P}_B(\mathcal{D}, K^c)$, we first form the digraph $\mathcal{D}^*$ shown in Fig. 4.12 , by condensing the strong components of $\mathcal{D}$, and rename the feedback edges as $(k_{11}, k_{12}, k_{21}, k_{22}) = (e_1, e_2, e_3, e_4)$.

The Implicit Enumeration Algorithm proceeds as follows:

0. $J = (F, F, F, F)$, $c^* = \gamma$.
1. $J = (1, F, F, F)$.
2. Test for feasibility of J:

$$\mathbf{W} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Since gr($\mathbf{W}$)=3 $< 4=N$, J is not feasible.

3. No such l exists.
4. Test for potential feasibility of J: Since J contains only 1 and F, $\mathbf{W}$ is full. Therefore, J is potentially feasible.
1. $J = (1, 1, F, F)$.
2. J is feasible. $J^* = (1, 1, F, F)$, $c^* = c(J) = 6$.
5. $J = (1, 0, F, F)$.
3. No such l exists.
4. J is potentially feasible.
1. $J = (1, 0, 1, F)$.
2. J is not feasible.
3. As $c(J) + c_4 = 6 = c^*$, $J = (1, 0, 1, U)$. No l with $J(l) = F$ remains.
5. $J = (1, 0, 0, F)$.
3. No such l exists
4. J is potentially feasible.
1. $J = (1, 0, 0, 1)$.
2. J is feasible. $J^* = (1, 0, 0, 1)$, $c^* = c(J) = 5$.
5. $J = (1, 0, 0, 0)$
3. No such l exists. No l with $J(l) = F$ remains.
5. $J = (0, F, F, F)$.
3. No such l exists.
4. J is potentially feasible.
1. $J = (0, 1, F, F)$.
2. J is feasible . $J^* = (0, 1, F, F)$, $c^* = c(J) = 4$.
5. $J = (0, 0, F, F)$.
3. No such l exists.
4. J is not potentially feasible.
5. $J(l) \neq 1$ for all l.
6.

$$\mathbf{K}_B^o = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad c(\mathbf{K}_B^o) = c^* = 4.$$

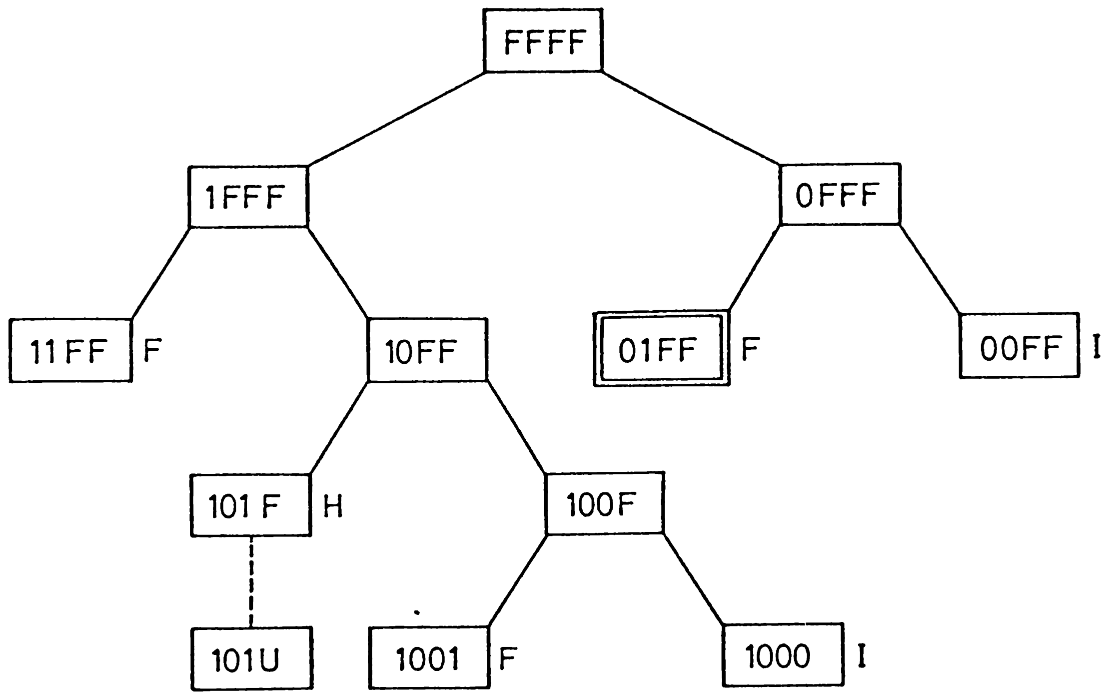


Figure 4.13: The tree generated by the implicit enumeration algorithm

The tree generated by the implicit enumeration algorithm is shown in Fig. 4.13, where the branches are terminated at the nodes marked as F, H or I. An F indicates that the corresponding pattern is a feasible one better than the previous feasible, an H marks a pattern whose cost is higher than the cost of the current best pattern (even if it is potentially feasible), and an I marks the infeasible patterns.

The suboptimal solution to problem $\mathcal{P}(\mathcal{D}, K^c)$ is obtained by combining $\mathbf{K}_A^o$ and $\mathbf{K}_B^o$ as

$$\mathbf{K}^s = \mathbf{K}_A^o + \mathbf{K}_B^o = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

resulting in $c(\mathbf{K}^s) = 6$.

Now considering the reachability matrix of $\mathcal{D}^*$, we observe that

$$\mathbf{G}^* = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{H}^* = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

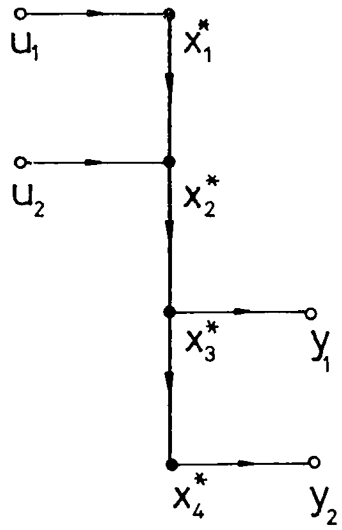


Figure 4.14: The intermediate digraph corresponding to the digraph of Fig. 4.12

which already have the special forms in (46), with $s_1 = 1$, $s_2 = 2$; $t_1 = 3$, $t_2 = 4$. The intermediate digraph $\mathcal{D}_I^*$ is shown in Fig. 4.14, and $K_M^c = K^c$.

It can easily be verified that application of the implicit enumeration algorithm to $\mathcal{P}_B(\mathcal{D}_I^*, K_M^c)$ yields the same optimal feedback pattern $\mathbf{K}_B^o$.

To illustrate construction of $\mathcal{D}_M$, we compute

$k_{in,2} = k_{in,3} = k_{out,2} = k_{out,3} = 2$,

and expand x_2^* and x_3^* as shown in Figs. 4.15, and 4.16.

Application of the Hungarian Assignment Algorithm to the modified system cost matrix $\hat{\mathcal{S}}_M^c$ yields the same optimum solution $\mathbf{K}_B^o$.

If we employ the sequential optimization procedure, the cost matrix K^c is modified to

$$K_A^c = \begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix}$$

after solving $\mathcal{P}_A(\mathcal{D}, K^c)$. Now, the optimum solution of $\mathcal{P}_B(\mathcal{D}, K_A^c)$ is obtained, either by the implicit enumeration algorithm or through the use of the modified digraph, to be

$$\acute{\mathbf{K}}_B^o = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}.$$

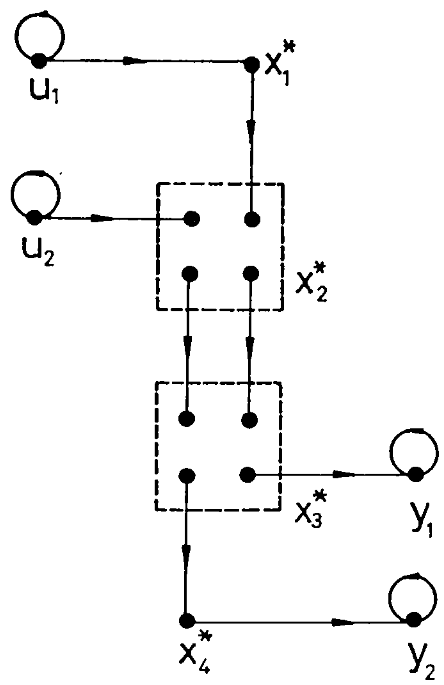


Figure 4.15: The modified digraph of Example 4.2

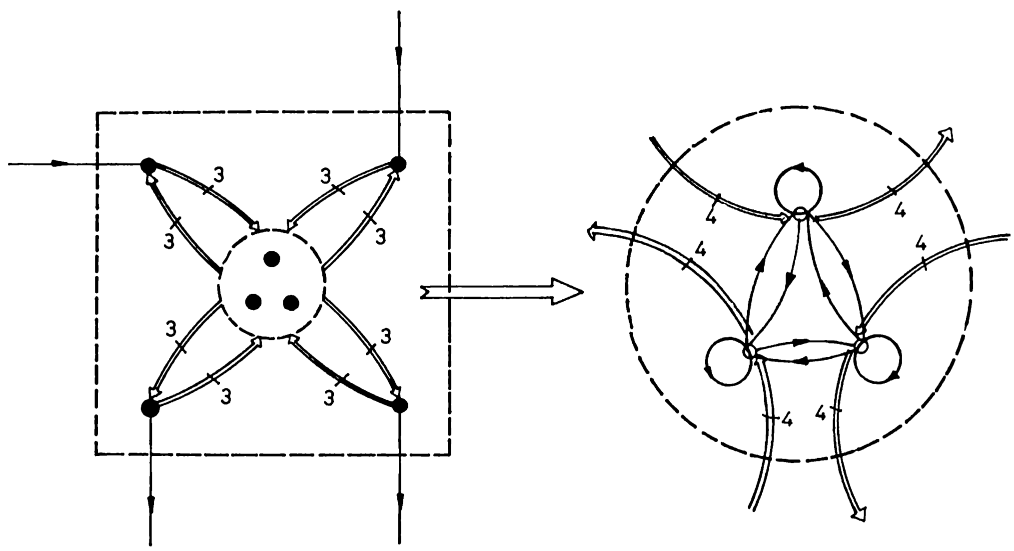


Figure 4.16: Expansion of x_2^*

The corresponding suboptimal solution will then be

$$\mathbf{K}^s_{AB} = \mathbf{K}^o_A + \acute{\mathbf{K}}^o_B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

resulting in $c(\mathbf{K}^s_{AB}) = 5$.

Finally, we point out that if the sequential optimization procedure is employed in the sequence $(\mathcal{P}_B, \mathcal{P}_A)$, then we obtain

$$\mathbf{K}^o_B = \acute{\mathbf{K}}^o_A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

so that $\mathbf{K}^s_{BA} = \mathbf{K}^o$, that is the suboptimal solution coincides with the optimal one.

5. CONCLUSIONS

The problem of identifying a minimum cost feedback pattern, which does not give rise to structurally fixed modes, is considered. The problem is formulated in a graph-theoretic setting, and the graphical characterization of the fixed modes is utilized as the basic tool. A classification of the structurally fixed modes into two distinct types allows a decomposition of the problem into two simpler subproblems, whose optimum solutions can be combined to obtain a suboptimal solution to the original problem. These two subproblems are reformulated as network flow problems , and concepts from network theory are utilized to obtain their solutions.

Several remarks can be made concerning the formulation and the solution of the problem. First, it is observed that the problem of choosing a feasible feedback pattern that includes a minimum number of feedback edges, which was considered previously in [19], is a special case of the problem formulated in this work, which corresponds to the case k_{ij}^c=1 for all i, j for which $k_{ij}^c \neq \gamma$. However still more general formulations are possible. For example, fixed initial costs can be assigned to the inputs and outputs in addition to the feedback costs. It may also be meaningful to group the inputs and outputs as in decentralized control, and assign costs to the multiple feedback links among the groups rather than to the individual links. These complications, however , make the already nonlinear problem even more difficult to handle.

As a final remark for this work we emphasize that the subproblems to which the main problem is decomposed, have different natures in the sense of computational complexity. Problem $\mathcal{P}_A$, being an *Assignment problem*, can be solved by polynomial time algorithms [20],[28], whereas $\mathcal{P}_B$, being a *Generalized assignment problem*, has been shown to be unsolvable by polynomial time algorithms in general.

REFERENCES

[1] Y. C. Ho and S. K. Mitter. *Directions in Large Scale Systems.* Plenum, 1976.

[2] M. S. Mahmoud. Multilevel systems control and applications: a survey. *IEEE Trans. Sys. Man. Cyb.*, 7:125–143, 1977.

[3] M. Athans, M. G. Safanov, N. R. Sandell Jr. and P. Varaiya. Survey of decentralized control methods for large-scale systems. *IEEE Trans. Auto. Cont. (special issue on large-scale systems)*, 23:108–128, 1978.

[4] M. Jamshidi. *Large Scale Systems: Modeling and Control.* North Holland, 1983.

[5] O. I. Elgerd. *Electric Energy Systems Theory: An Introduction.* McGraw-Hill, 1971.

[6] E. C. Tacker, C. W. Sanders and T. D. Linton. A new class of decentralized filters for interconnected systems. *IEEE Trans. Auto. Cont.*, 19:259–262, 1974.

[7] A. Gersho and B. J. Karafin. Mutual synchronization of geographically seperated oscillators. *Bell Systems Tech. Jour.*, 1689–1701, 1966.

[8] I. W. Sandberg. On conditions under which it is possible to synchronize digital transmission systems. *Bell Systems Tech. Jour.*, 46(6):1999–2020, 1969.

[9] M Aoki and M. T. Li. Partial reconstruction of state vectors in decentralized dynamic systems. *IEEE Trans. Auto. Cont.*, 18:289–294, 1973.

[10] S. H. Wang and E. J. Davison. On the stabilization of large scale systems. *IEEE Trans. Auto. Cont.*, 11:473–478, 1978.

[11] J. F. Corfmat and A. S. Morse. Decentralized control of linear multivariable systems. *Automatica*, 11:479–495, 1976.

[12] B. D. O. Anderson and D. J. Clements. Algebraic characterization of fixed modes in decentralized control. *Automatica*, 17:703–712, 1981.

[13] C. -T. Lin. Structural controllability. *IEEE Trans. Auto. Cont.*, 19:201–208, 1974.

[14] R. W. Shields and J. B. Pearson. Structural controllability of multi-input linear systems. *IEEE Trans. Auto. Cont.*, 21:203–212, 1976.

[15] K. Glover and L. M. Silverman. Characterization of structural controllability. *IEEE Trans. Auto. Control*, 21:534–537, 1976.

[16] M. E. Sezer and D. D. Siljak. Structurally fixed modes. *Systems and Control Letters*, 1:60–64, 1981.

[17] M. E. Sezer, V. Pichai and D. D. Siljak. A graph-theoretic characterization of structurally fixed modes. *Automatica*, 20:247–250, 1984.

[18] K. Reinschke. Graph theoretic characterization of fixed modes in centralized and decentralized control. *Int. J. Control*, 39:715–729, 1984.

[19] M. E. Sezer. Minimal essential feedback patterns for pole assignment using dynamic compensators. In *Proc. 22nd IEEE Conf. on Decision and Control*, 1983.

[20] M. S. Bazaraa and J. J. Jarvis. *Linear Programming and Network Flows.* Wiley, 1977.

[21] J. Moore, V. Chaera and P. M. Ghare. *Applications of Graph Theory Algorithms.* North Holland, 1977.

[22] R. Z. Norman, F. Harary and D. Cartwright. *Structural Models: An Introduction to the Theory of Directed Graphs.* Wiley, 1965.

[23] N. Deo. *Graph Theory with Applications to Engineering and Computer Science.* Prentice Hall, 1974.

[24] C. L. Coates. Flow graph solutions of linear graph algorithms. *IRE Trans. Circ. Theory*, 6:170–187, 1959.

[25] D. D. Siljak. *Large Scale Dynamic Systems: Stability and Structure.* North Holland, 1978.

[26] I. S. Duff. On algorithms for obtaining a maximum transversal. *ACM Trans. Math. Software*, 7(3):315–330, 1981.

[27] C. H. Papadimitriou and J. Tsitsiklis. A simple criterion for structurally fixed modes. *Systems and Control Letters*, 4:333–337, 1984.

[28] H. W. Kuhn. The hungarian method for the assignment problem. *Naval Research Logistics Quarterly*, 2:83–97, 1955.

[29] R. Garfinkel and G. Nemhauser. *Integer Programming.* Wiley, 1972.

[30] R. Tarjan. Depth first search and linear graph algorithms. *SIAM J. Comput.*, 1:146–160, 1972.

APPENDIX. An algorithm which computes the generic rank of a structure matrix

I. Initial Assignment

a) Pick an arbitray element in A. Delete the row and column at which the element picked is located.

b) Repeat (a) until no more selection is possible. Let the number of elements picked be r.

c) If $r = min\ \{p, q\}$, stop! $gr(A) = r$. Otherwise, proceed to II.

II. Permutation

a) Permute A into

$$\begin{array}{cc} \tilde{A} = \begin{bmatrix} F & G \\ 0 & H \end{bmatrix} & \begin{array}{l} r \\ p-r \end{array} \\ \begin{array}{cc} \quad\quad\ \mathrm{q-r} & \mathrm{r} \end{array} & \end{array}$$

where G has nonzero elements in its main digonal.

b) If $F = 0$ or $H = 0$, stop! $gr(A) = r$. Otherwise, proceed to III.

III. Increase r

a) Permute $\tilde{A}$ into

$$\tilde{\tilde{A}} = \underbrace{\begin{bmatrix} F_1 & G_{11} & G_{12} & \dots & G_{1k} \\ 0 & F_2 & G_{22} & \dots & G_{2k} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & & F_k & G_{kk} \\ 0 & 0 & \dots & 0 & \tilde{H} \end{bmatrix}}_{\mathrm{q-r}\qquad\qquad \mathrm{r}} \begin{array}{l} \left.\vphantom{\begin{matrix} a\\a\\a\\a \end{matrix}}\right\} r \\ \left.\vphantom{a}\right\} p-r \end{array}$$

where G_{ii} are square and have nonzero elements on their main diagonals, $i = 1, 2, ..., k$; F_i have no zero rows, $i = 1, 2, ..., k-1$; and F_k is either zero, or has no zero rows. (If F has no zero rows to start with, then $\tilde{A}$ is already in the form of $\tilde{\tilde{A}}$ with k=1, $F_1 = F$, $G_{11} = G$, $\tilde{H} = H$.)

b) If F_k=0, stop! $gr(A) = r$. Otherwise proceed to (c).

c) Construct a chain starting from a nonzero entry of $\tilde{H}$, including one nonzero entry from each F_i, G_{ii}, $i = 1, ..., k$, such that

$$\tilde{H} \to G_{kk} \to F_k \to G_{k-1,k-1} \to F_{k-1} \to \dots \to F_2 \to G_{11} \to F_1,$$

and replace the assigned entries of G_i, $i = 1, ..., k$, in this chain with those of $\tilde{H}$, and F_i, $i = 1, ..., k$. Hence increase the assignment by one, set $r = r + 1$, go to II.